Průběh funkce

Robert Mařík

27. června 2006

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Obsah

$y = \frac{x}{1+x^2}$ 3

$y = \frac{3x+1}{x^3}$ 49

$y = \frac{2(x^2-x+1)}{(x-1)^2}$ 101

$y = \frac{x^3}{3-x^2}$ 149

$y = \frac{x^2+1}{x^2-1}$ 191

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$$\boxed{y = \frac{x}{1+x^2}}$$ $D(f) = \mathbb{R}$;

- Omezení na definiční obor vyplývá ze jmenovatele zlomku.
- Výraz $x^2 + 1$ nesmí být nulový.
- To je však zajištěno pro všechna reálná čísla.

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$\boxed{y = \dfrac{x}{1+x^2}}$ $D(f) = \mathbb{R}$; lichá;

- Čitatel, x, je lichá funkce, jmenovatel, $(1 + x^2)$, je funkce sudá.
- Jako celek je tedy zlomek lichá funkce.

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$\boxed{y = \dfrac{x}{1+x^2}}$ $D(f) = \mathbb{R}$; lichá;

$y = 0$

Určíme průsečík s osou x a znaménko funkce na jednotlivých intervalech.

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$$\boxed{y = \frac{x}{1+x^2}}$$ $D(f) = \mathbb{R}$; lichá;

$$y = 0 \Rightarrow \frac{x}{1+x^2} = 0$$

Určíme průsečík s osou x a znaménko funkce na jednotlivých intervalech.

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$\boxed{y = \frac{x}{1+x^2}}$ $D(f) = \mathbb{R}$; lichá;

$$y = 0 \Rightarrow \frac{x}{1+x^2} = 0 \Rightarrow x = 0$$

Zlomek je roven nule právě tehdy, když čitatel je nulový.

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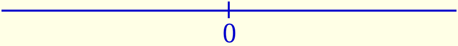
$\boxed{y = \dfrac{x}{1+x^2}}$ $D(f) = \mathbb{R}$; lichá;

$$y = 0 \Rightarrow \frac{x}{1+x^2} = 0 \Rightarrow x = 0$$



Zakreslíme průsečík $x = 0$ na osu x. Funkce nemá žádný bod nespojitosti.

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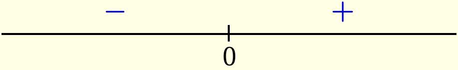
$\boxed{y=\dfrac{x}{1+x^2}}$ $D(f)=\mathbb{R}$; lichá;

$$y=0 \Rightarrow \frac{x}{1+x^2}=0 \Rightarrow x=0$$



- Jmenovatel $(1+x^2)$ je stále kladný.
- Čitatel zlomku má proto stejné znaménko jako celý zlomek $\dfrac{x}{1+x^2}$.
- Funkce je kladná, je-li x kladné a naopak.

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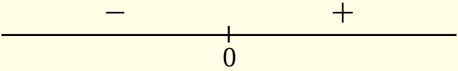
$\boxed{y = \dfrac{x}{1+x^2}}$ $D(f) = \mathbb{R}$; lichá;

$$y = 0 \Rightarrow \frac{x}{1+x^2} = 0 \Rightarrow x = 0$$



$$\lim_{x\to\pm\infty} \frac{x}{1+x^2}$$

Určíme limity v nekonečnu.

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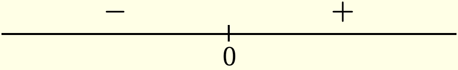
$\boxed{y = \dfrac{x}{1+x^2}}$ $D(f) = \mathbb{R}$; lichá;

$$y = 0 \Rightarrow \frac{x}{1+x^2} = 0 \Rightarrow x = 0$$



$$\lim_{x\to\pm\infty} \frac{x}{1+x^2} = \lim_{x\to\pm\infty} \frac{1}{x}$$

- Víme, že o výsledku rozhodují jenom vedoucí členy v čitateli a ve jmenovateli.
- Zelenou část lze vynechat.
- Zbytek zkrátíme: $\dfrac{x}{x^2} = \dfrac{1}{x}$.

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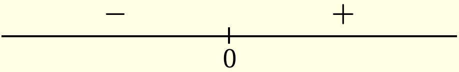
$\boxed{y = \frac{x}{1+x^2}}$ $D(f) = \mathbb{R}$; lichá;

$$y = 0 \Rightarrow \frac{x}{1+x^2} = 0 \Rightarrow x = 0$$



$$\lim_{x\to\pm\infty} \frac{x}{1+x^2} = \lim_{x\to\pm\infty} \frac{1}{x} = \frac{1}{\pm\infty}$$

Dosadíme.

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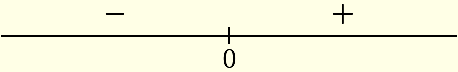
$\boxed{y=\dfrac{x}{1+x^2}}$ $D(f)=\mathbb{R}$; lichá;

$$y=0 \Rightarrow \frac{x}{1+x^2}=0 \Rightarrow x=0$$



$$\lim_{x\to\pm\infty}\frac{x}{1+x^2}=\lim_{x\to\pm\infty}\frac{1}{x}=\frac{1}{\pm\infty}=0$$

- Obě hodnoty $\frac{1}{\infty}$ i $\frac{1}{-\infty}$ jsou nulové.
- Funkce má vodorovnou asymptotu $y=0$ pro x jdoucí k $\pm\infty$.

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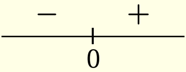
$\boxed{y = \dfrac{x}{1+x^2}}$ $D(f) = \mathbb{R}$; lichá;



$$y' = \frac{1(1+x^2) - x(0+2x)}{(1+x^2)^2}$$

- Vypočteme derivaci.
- Derivujeme podíl podle vzorce pro derivaci podílu.

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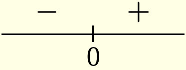
$$\boxed{y = \frac{x}{1+x^2}}$$ $D(f) = \mathbb{R}$; lichá;



$$y' = \frac{1(1+x^2) - x(0+2x)}{(1+x^2)^2}$$
$$= \frac{1+x^2-2x^2}{(1+x^2)^2}$$

Upravíme.

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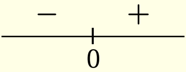
$$\boxed{y=\frac{x}{1+x^2}}$$ $D(f)=\mathbb{R}$; lichá;



$$
\begin{aligned}
y' &= \frac{1(1+x^2)-x(0+2x)}{(1+x^2)^2}\\
&= \frac{1+x^2-2x^2}{(1+x^2)^2}\\
&= \frac{1-x^2}{(1+x^2)^2}
\end{aligned}
$$

Upravíme.

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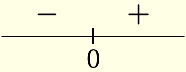
$$\boxed{y = \frac{x}{1+x^2}}$$ $D(f) = \mathbb{R}$; lichá;



$$y' = \frac{1-x^2}{(1+x^2)^2}\ ;$$

$$y' = 0$$

Hledáme řešení rovnice $y' = 0$.

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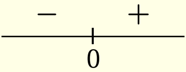
$\boxed{y=\frac{x}{1+x^2}}$ $D(f)=\mathbb{R}$; lichá;



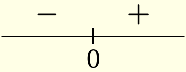
$y'=\frac{1-x^2}{(1+x^2)^2}$;

$$y'=0$$

$$\frac{1-x^2}{(1+x^2)^2}=0$$

Dosadíme za derivaci.

$\boxed{y = \dfrac{x}{1+x^2}}$ $D(f) = \mathbb{R}$; lichá;



$$y' = \frac{1-x^2}{(1+x^2)^2} \;;$$

$$\begin{aligned} y' &= 0 \\ \frac{1-x^2}{(1+x^2)^2} &= 0 \\ 1-x^2 &= 0 \end{aligned}$$

Zlomek je nulový, má-li nulový čitatel.

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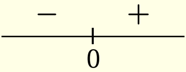
$\boxed{y = \frac{x}{1+x^2}}$ $D(f) = \mathbb{R}$; lichá;



$y' = \frac{1-x^2}{(1+x^2)^2}$;

$$\begin{aligned} y' &= 0 \\ \frac{1-x^2}{(1+x^2)^2} &= 0 \\ 1-x^2 &= 0 \\ x^2 &= 1 \end{aligned}$$

Vyjádříme x^2.

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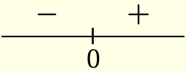
$$\boxed{y = \frac{x}{1+x^2}}$$ $D(f) = \mathbb{R}$; lichá;



$$y' = \frac{1-x^2}{(1+x^2)^2}\ ;$$

$$\begin{aligned} y' &= 0 \\ \frac{1-x^2}{(1+x^2)^2} &= 0 \\ 1-x^2 &= 0 \\ x^2 &= 1 \\ x_1 &= 1 \\ x_2 &= -1 \end{aligned}$$

Vypočítáme x. Dostáváme dvě řešení.

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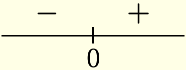
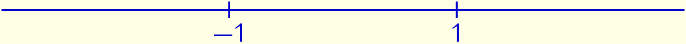
$$\boxed{y = \frac{x}{1+x^2}}$$ $D(f) = \mathbb{R}$; lichá;



$$y' = \frac{1-x^2}{(1+x^2)^2}\ ; \qquad x_{1,2} = \pm 1$$



- Nakreslíme osu x a stacionární body.
- Nejsou žádné body nespojitosti.

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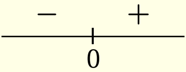
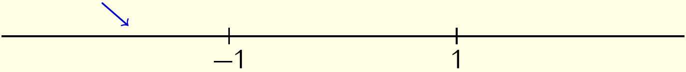
$\boxed{y = \dfrac{x}{1+x^2}}$ $D(f) = \mathbb{R}$; lichá;



$y' = \dfrac{1-x^2}{(1+x^2)^2}$; $x_{1,2} = \pm 1$



Testujeme $x = -2$. Dostáváme

$$y'(-2) = \frac{1-4}{\text{kladná hodnota}} < 0.$$

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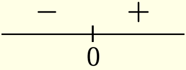
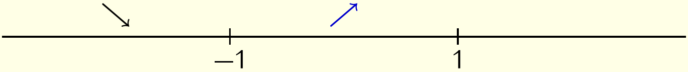
$$\boxed{y = \frac{x}{1+x^2}}$$ $D(f) = \mathbb{R}$; lichá;



$$y' = \frac{1-x^2}{(1+x^2)^2}\ ; \qquad x_{1,2} = \pm 1$$



Testujeme $x = 0$.

$$y'(0) = \frac{1}{1} > 0$$

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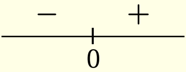
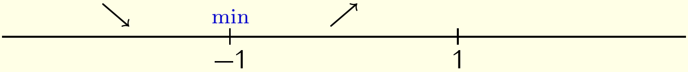
$$\boxed{y = \frac{x}{1+x^2}}$$ $D(f) = \mathbb{R}$; lichá;



$$y' = \frac{1-x^2}{(1+x^2)^2}\ ; \qquad x_{1,2} = \pm 1$$



Funkce má lokální minimum v bodě $x = -1$. Funkční hodnota je

$$y(-1) = \frac{-1}{1+(-1)^2} = -\frac{1}{2}.$$

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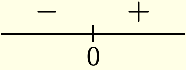
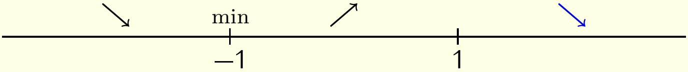
$\boxed{y = \frac{x}{1+x^2}}$ $D(f) = \mathbb{R}$; lichá;



$y' = \frac{1-x^2}{(1+x^2)^2}$; $x_{1,2} = \pm 1$



Testujeme $x = 2$. Platí

$$y'(2) = \frac{1-4}{\text{kladná hodnota}} < 0.$$

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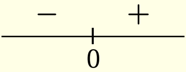
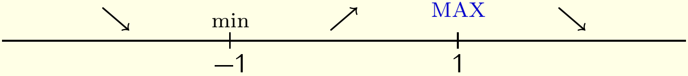
$$\boxed{y = \frac{x}{1+x^2}}$$ $D(f) = \mathbb{R}$; lichá;



$$y' = \frac{1-x^2}{(1+x^2)^2}\ ; \qquad x_{1,2} = \pm 1$$



Funkce má lokální maximum v bodě $x = 1$. Funkční hodnota je

$$y(1) = -y(-1) = \frac{1}{2},$$

kde jsme využili toho, že funkce je lichá a hodnota $y(-1)$ již byla vypočítána.

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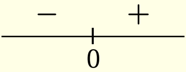
$\boxed{y = \dfrac{x}{1+x^2}}$ $D(f) = \mathbb{R}$; lichá;



$$y' = \frac{1-x^2}{(1+x^2)^2}\ ; \qquad x_{1,2} = \pm 1$$

$$y'' = \left(\frac{1-x^2}{(1+x^2)^2}\right)'$$

Vypočteme druhou derivaci.

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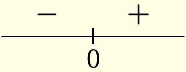
$$\boxed{y = \frac{x}{1+x^2}}$$ $D(f) = \mathbb{R}$; lichá;



$$y' = \frac{1-x^2}{(1+x^2)^2}\ ; \qquad x_{1,2} = \pm 1$$

$$y'' = \left(\frac{1-x^2}{(1+x^2)^2}\right)'$$

$$= \frac{-2x(1+x^2)^2 - (1-x^2)2(1+x^2)(0+2x)}{(1+x^2)^4}$$

- Derivuje podíl podle vzorce pro derivaci podílu.
- Jmenovatel derivujeme jako složenou funkci. Tím se nezbavíme možnosti vytknout v čitateli a zkrátit zlomek.

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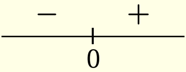
$$\boxed{y=\frac{x}{1+x^2}}$$ $D(f)=\mathbb{R}$; lichá;



$$y'=\frac{1-x^2}{(1+x^2)^2}\ ; \qquad x_{1,2}=\pm 1$$

$$\begin{aligned}
y''&=\left(\frac{1-x^2}{(1+x^2)^2}\right)'\\
&=\frac{-2x(1+x^2)^2-(1-x^2)2(1+x^2)(0+2x)}{(1+x^2)^4}\\
&=\frac{-2x(1+x^2)[(1+x^2)+(1-x^2)2]}{(1+x^2)^4}
\end{aligned}$$

Vytkneme

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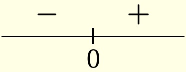
$\boxed{y = \dfrac{x}{1+x^2}}$ $D(f) = \mathbb{R}$; lichá;



$$y' = \frac{1-x^2}{(1+x^2)^2}\ ; \qquad x_{1,2} = \pm 1$$

$$\begin{aligned}
y'' &= \left(\frac{1-x^2}{(1+x^2)^2}\right)' \\
&= \frac{-2x(1+x^2)^2 - (1-x^2)2(1+x^2)(0+2x)}{(1+x^2)^4} \\
&= \frac{-2x(1+x^2)[\ 1+x^2\ \ + (1-x^2)2]}{(1+x^2)^4} \\
&= \frac{-2x[3-x^2]}{(1+x^2)^3}
\end{aligned}$$

Zelené části se zkrátí. Zjednodušíme výraz v hranaté závorce.

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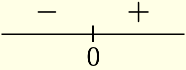
$\boxed{y = \frac{x}{1+x^2}}$ $D(f) = \mathbb{R}$; lichá;



$$y' = \frac{1-x^2}{(1+x^2)^2}\;; \qquad x_{1,2} = \pm 1$$

$$\begin{aligned}
y'' &= \left(\frac{1-x^2}{(1+x^2)^2}\right)' \\
&= \frac{-2x(1+x^2)^2 - (1-x^2)2(1+x^2)(0+2x)}{(1+x^2)^4} \\
&= \frac{-2x(1+x^2)[\;1+x^2\;\; + (1-x^2)2]}{(1+x^2)^4} \\
&= \frac{-2x[3-x^2]}{(1+x^2)^3} \\
&= 2\frac{x(x^2-3)}{(1+x^2)^3}
\end{aligned}$$

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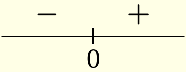
$\boxed{y = \dfrac{x}{1+x^2}}$ $D(f) = \mathbb{R}$; lichá;



$$y' = \frac{1-x^2}{(1+x^2)^2}\ ; \qquad x_{1,2} = \pm 1$$

$$y'' = 2\frac{x(x^2-3)}{(1+x^2)^3} \quad \Rightarrow \quad 2\frac{x(x^2-3)}{(1+x^2)^3} = 0$$

Vyřešíme $y'' = 0$.

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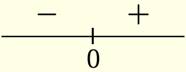
$\boxed{y = \dfrac{x}{1+x^2}}$ $D(f) = \mathbb{R}$; lichá;



$$y' = \frac{1-x^2}{(1+x^2)^2}\ ; \qquad x_{1,2} = \pm 1$$

$$y'' = 2\frac{x(x^2-3)}{(1+x^2)^3} \quad \Rightarrow \quad 2\frac{x(x^2-3)}{(1+x^2)^3} = 0 \quad \Rightarrow \quad x(x^2-3) = 0$$

Zlomek je nulový, je-li nulový jeho čitatel.

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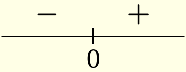
$\boxed{y = \dfrac{x}{1+x^2}}$ $D(f) = \mathbb{R}$; lichá;



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$x_3 = 0,\ x_4 = \sqrt{3},\ x_5 = -\sqrt{3}$

Jsou dvě možnosti: buď $x = 0$, nebo $x^2 - 3 = 0$. Druhá z možností vede na rovnici

$$x^2 = 3$$
$$x = \pm\sqrt{3}.$$

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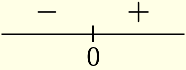
$\boxed{y = \dfrac{x}{1+x^2}}$ $D(f) = \mathbb{R}$; lichá;



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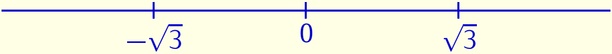

Vyznačíme body na osu x. Nejsou zde žádné body nespojitosti.

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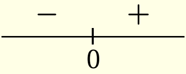
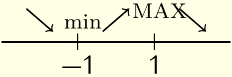
$$\boxed{y = \frac{x}{1+x^2}}$$ $D(f) = \mathbb{R}$; lichá;



$$y' = \frac{1-x^2}{(1+x^2)^2}\ ; \qquad x_{1,2} = \pm 1$$



$$y'' = 2\frac{x(x^2-3)}{(1+x^2)^3} \quad\Rightarrow\quad 2\frac{x(x^2-3)}{(1+x^2)^3} = 0 \quad\Rightarrow\quad x(x^2-3) = 0$$

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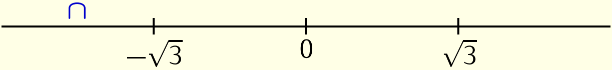
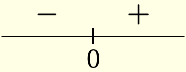


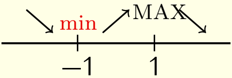
Testujeme $x = -2$.

$$y''(-2) = 2\frac{-2(4-3)}{\text{kladná hodnota}} < 0.$$

$$\boxed{y=\frac{x}{1+x^2}}$$ $D(f)=\mathbb{R}$; lichá;

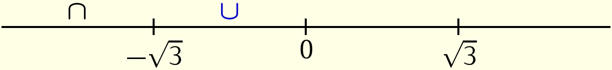


$$y'=\frac{1-x^2}{(1+x^2)^2}\ ;\qquad x_{1,2}=\pm 1$$



$$y''=2\frac{x(x^2-3)}{(1+x^2)^3}\quad\Rightarrow\quad 2\frac{x(x^2-3)}{(1+x^2)^3}=0\quad\Rightarrow\quad x(x^2-3)=0$$

$$x_3=0,\ x_4=\sqrt{3},\ x_5=-\sqrt{3}$$



Testujeme $x=-1$. Funkce je v tomto bodě konvexní, protože je zde lokální minimum.

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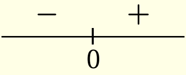
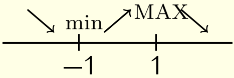
$\boxed{y = \dfrac{x}{1+x^2}}$ $D(f) = \mathbb{R}$; lichá;

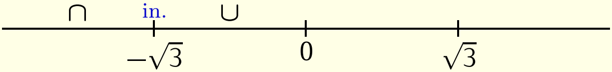


$y' = \dfrac{1-x^2}{(1+x^2)^2}$; $x_{1,2} = \pm 1$



$$y'' = 2\frac{x(x^2-3)}{(1+x^2)^3} \quad \Rightarrow \quad 2\frac{x(x^2-3)}{(1+x^2)^3} = 0 \quad \Rightarrow \quad x(x^2-3) = 0$$

$x_3 = 0,\ x_4 = \sqrt{3},\ x_5 = -\sqrt{3}$



V bodě $x = -\sqrt{3}$ je inflexe. Funkční hodnota je

$$y(-\sqrt{3}) = \frac{-\sqrt{3}}{1+3} \approx -0.43.$$

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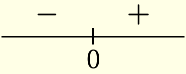
$$\boxed{y = \frac{x}{1+x^2}}$$ $D(f) = \mathbb{R}$; lichá;



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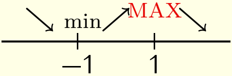
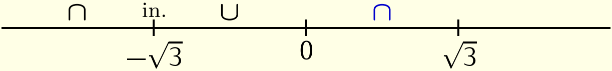


$$y'' = 2\frac{x(x^2-3)}{(1+x^2)^3} \quad \Rightarrow \quad 2\frac{x(x^2-3)}{(1+x^2)^3} = 0 \quad \Rightarrow \quad x(x^2-3) = 0$$

$x_3 = 0,\ x_4 = \sqrt{3},\ x_5 = -\sqrt{3}$



Testujeme $x = 1$. Funkce je v tomto bodě konkávní, protože je zde lokální maximum.

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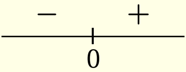
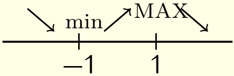
$\boxed{y = \frac{x}{1+x^2}}$ $D(f) = \mathbb{R}$; lichá;



$y' = \frac{1-x^2}{(1+x^2)^2}$; $\quad x_{1,2} = \pm 1$



$$y'' = 2\frac{x(x^2-3)}{(1+x^2)^3} \quad \Rightarrow \quad 2\frac{x(x^2-3)}{(1+x^2)^3} = 0 \quad \Rightarrow \quad x(x^2-3) = 0$$

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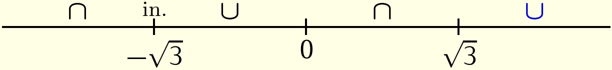

Testujeme $x = 2$. Dostáváme

$$y''(2) = 2\frac{2(4-3)}{\text{něco kladného}} > 0.$$

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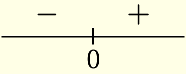
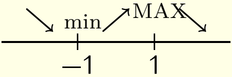
$$\boxed{y=\frac{x}{1+x^2}}$$ $D(f)=\mathbb{R}$; lichá;

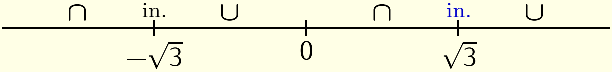


$$y'=\frac{1-x^2}{(1+x^2)^2}\ ; \qquad x_{1,2}=\pm 1$$



$$y''=2\frac{x(x^2-3)}{(1+x^2)^3} \quad\Rightarrow\quad 2\frac{x(x^2-3)}{(1+x^2)^3}=0 \quad\Rightarrow\quad x(x^2-3)=0$$

$x_3=0,\ x_4=\sqrt{3},\ x_5=-\sqrt{3}$



Inflexe v bodě $x=\sqrt{3}$. Funkční hodnota je

$$y(\sqrt{3})=\frac{\sqrt{3}}{1+3}\approx 0.43.$$

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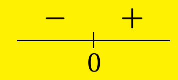

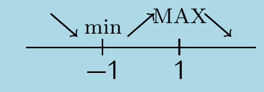

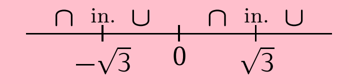

$f(0) = 0$
$f(\pm\infty) = 0$

$$f(\pm 1) = \pm\frac{1}{2}$$

$$f(\pm\sqrt{3}) \approx \pm 0.433$$

Vypíšeme si nejdůležitější výsledky.

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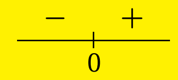

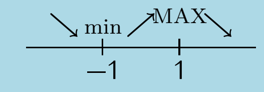

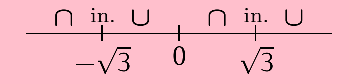

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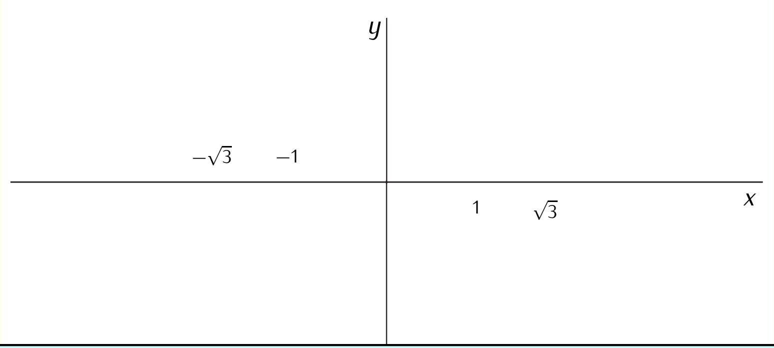

Zakreslíme souřadný systém.

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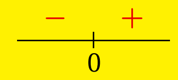

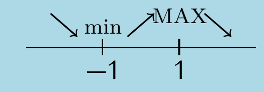

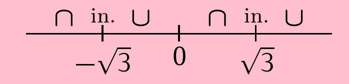

$f(0)=0$

$f(\pm\infty)=0$

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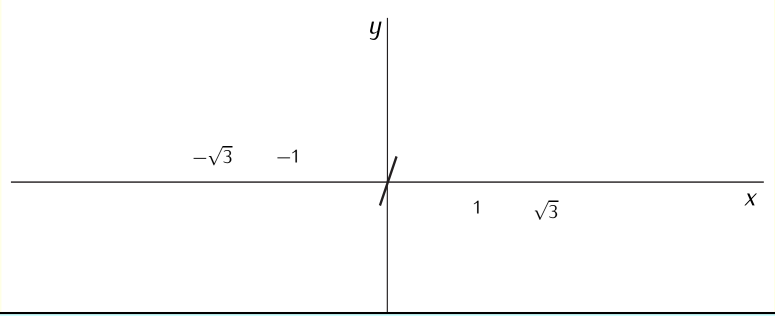
$$f(\pm\sqrt{3})\approx\pm 0.433$$



V bodě $x=0$ je průsečík s osou x. Funkční hodnoty se v tomto bodě mění z kladných na záporné.

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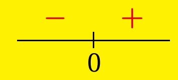

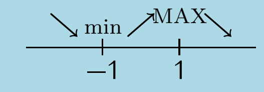

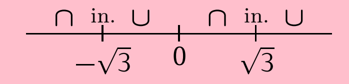

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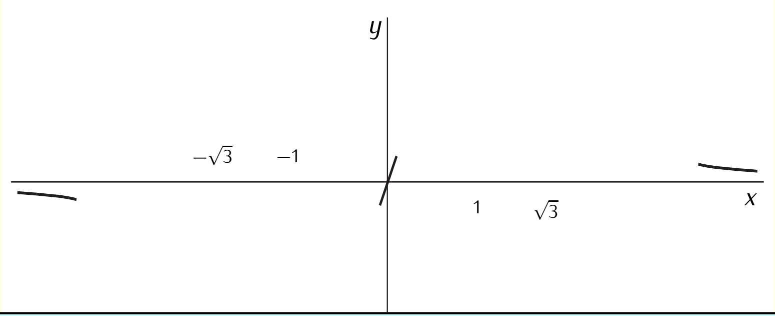

Zachytíme informaci o vodorovné tečně v $\pm\infty$. Dáváme si pozor na znaménko funkce, musíme graf správně nakreslit nad nebo pod asymptotu.

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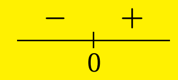

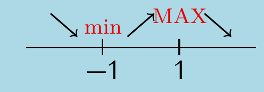

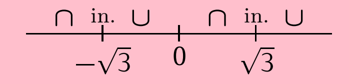

$f(0) = 0$
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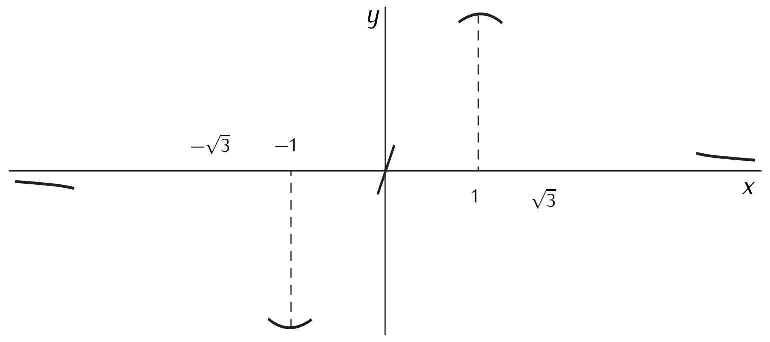

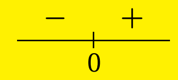

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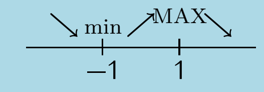

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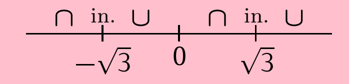

∩ in. ∪ ∩ in. ∪

$-\sqrt{3}$ 0 $\sqrt{3}$

$f(0) = 0$

$f(\pm\infty) = 0$

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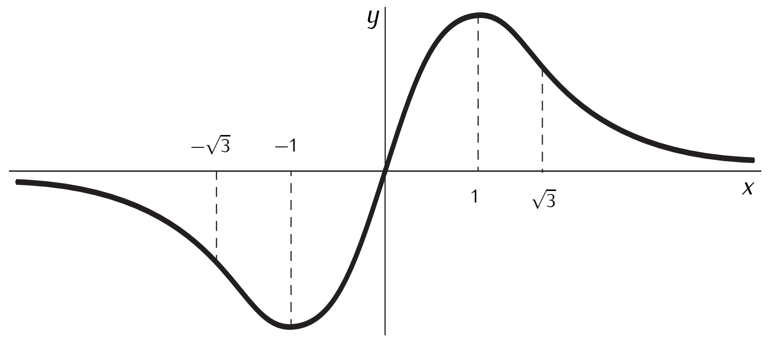

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$$y = \frac{3x+1}{x^3}$$

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$$\boxed{y = \frac{3x+1}{x^3}}$$ $D(f) = \mathbb{R} \setminus \{0\}$;

- Určíme definiční obor.
- Ve jmenovateli nesmí být nula.

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$$\boxed{y = \frac{3x+1}{x^3}} \quad D(f) = \mathbb{R} \setminus \{0\} \ ;$$

$$y = 0$$

Určíme průsečík s osou x jako řešení rovnice $y = 0$.

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$$\boxed{y = \frac{3x+1}{x^3}} \quad D(f) = \mathbb{R} \setminus \{0\}\ ;$$

$$y = 0$$

$$\frac{3x+1}{x^3} = 0$$

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$\boxed{y=\frac{3x+1}{x^3}}$ $D(f)=\mathbb{R}\setminus\{0\}$;

$$
\begin{aligned}
y &= 0\\
\frac{3x+1}{x^3} &= 0\\
3x+1 &= 0
\end{aligned}
$$

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$$\boxed{y = \frac{3x+1}{x^3}}$$ $D(f) = \mathbb{R} \setminus \{0\}$;

$$\begin{aligned} y &= 0 \\ \frac{3x+1}{x^3} &= 0 \\ 3x+1 &= 0 \\ x &= -\frac{1}{3} \end{aligned}$$

Funkce má s osou x jediný průsečík $x = -\frac{1}{3}$

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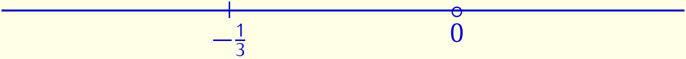
$\boxed{y = \dfrac{3x+1}{x^3}}$ $D(f) = \mathbb{R} \setminus \{0\}$;



- Určíme znaménka funkce.
- Rozdělíme osu x pomocí průsečíků a bodů nespojitosti na podintervaly, kde se znaménko zachovává.

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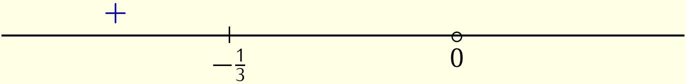
$\boxed{y=\dfrac{3x+1}{x^3}}$ $D(f)=\mathbb{R}\setminus\{0\}$;



Uvažujme interval zcela vlevo. Zvolme $x=-1$ a vypočteme

$$y(-1)=\frac{-3+1}{-1}=2>0.$$

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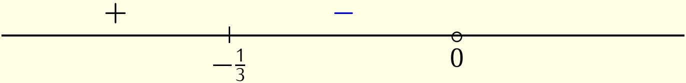
$\boxed{y = \dfrac{3x+1}{x^3}}$ $D(f) = \mathbb{R} \setminus \{0\}$;



Uvažujme prostřední interval, zvolme $x = -\dfrac{1}{4}$ a vypočteme

$$y(-\frac{1}{4}) = \frac{-\frac{3}{4}+1}{-\frac{1}{64}} = \frac{\frac{1}{4}}{-\frac{1}{64}} = -16 < 0.$$

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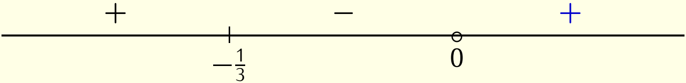
$\boxed{y = \dfrac{3x+1}{x^3}}$ $D(f) = \mathbb{R} \setminus \{0\}$;



V posledním intervalu zvolme $x = 1$ a vypočteme

$$y(1) = \frac{3+1}{1} = 4 > 0.$$

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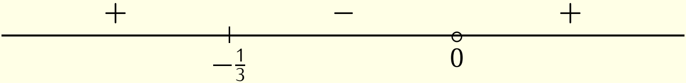
$$\boxed{y = \frac{3x+1}{x^3}} \quad D(f) = \mathbb{R} \setminus \{0\} \ ;$$



$$\lim_{x\to 0^+} \frac{3x+1}{x^3} =$$

$$\lim_{x\to 0^-} \frac{3x+1}{x^3} =$$

Najdeme jednostranné limity v bodech nespojitosti.

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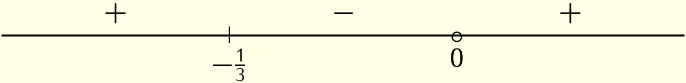
$$\boxed{y = \frac{3x+1}{x^3}} \quad D(f) = \mathbb{R} \setminus \{0\} \ ;$$



$$\lim_{x\to 0^+} \frac{3x+1}{x^3} = \frac{1}{0}$$

$$\lim_{x\to 0^-} \frac{3x+1}{x^3} = \frac{1}{0}$$

Dosazení $x = 0$ vede k výrazu typu $\frac{\text{nenulový výraz}}{\text{nula}}$.

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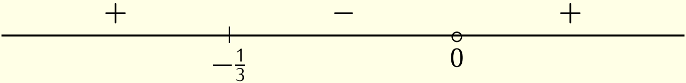
$\boxed{y = \frac{3x+1}{x^3}}$ $D(f) = \mathbb{R} \setminus \{0\}$;



$$\lim_{x\to 0^+} \frac{3x+1}{x^3} = \frac{1}{+0} = \infty$$
$$\lim_{x\to 0^-} \frac{3x+1}{x^3} = \frac{1}{-0} = -\infty$$

- Z přednášky víme, že jednostranné limity jsou nevlastní.
- Schéma se znaménkem funkce umožňuje odhalit, zda se funkce blíží k plus nebo minus nekonečnu.

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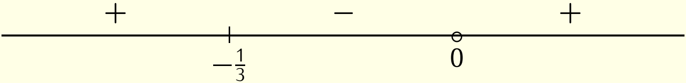
$$\boxed{y = \frac{3x+1}{x^3}} \quad D(f) = \mathbb{R} \setminus \{0\} \ ;$$



$$\lim_{x\to 0^+} \frac{3x+1}{x^3} = \frac{1}{+0} = \infty$$

$$\lim_{x\to 0^-} \frac{3x+1}{x^3} = \frac{1}{-0} = -\infty$$

$$\lim_{x\to \pm\infty} \frac{3x+1}{x^3}$$

Určíme limity v nevlastních bodech.

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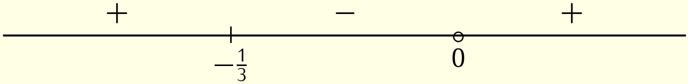
$$\boxed{y=\frac{3x+1}{x^3}}\quad D(f)=\mathbb{R}\setminus\{0\}\ ;$$



$$\lim_{x\to 0^+}\frac{3x+1}{x^3}=\frac{1}{+0}=\infty$$

$$\lim_{x\to 0^-}\frac{3x+1}{x^3}=\frac{1}{-0}=-\infty$$

$$\lim_{x\to\pm\infty}\frac{3x+1}{x^3}$$

Víme, že pouze vedoucí členy jsou podstatné v limitě tohoto typu a ostatní členy můžeme vynechat.

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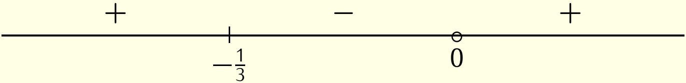
$$\boxed{y=\frac{3x+1}{x^3}}\quad D(f)=\mathbb{R}\setminus\{0\}\ ;$$



$$\lim_{x\to 0^+}\frac{3x+1}{x^3}=\frac{1}{+0}=\infty$$

$$\lim_{x\to 0^-}\frac{3x+1}{x^3}=\frac{1}{-0}=-\infty$$

$$\lim_{x\to\pm\infty}\frac{3x+1}{x^3}=\lim_{x\to\pm\infty}\frac{3}{x^2}$$

Zkrátíme x.

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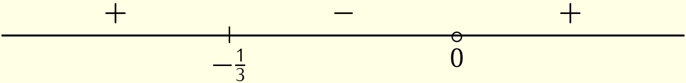
$\boxed{y = \frac{3x+1}{x^3}}$ $D(f) = \mathbb{R} \setminus \{0\}$;



$$\lim_{x\to 0^+} \frac{3x+1}{x^3} = \frac{1}{+0} = \infty$$

$$\lim_{x\to 0^-} \frac{3x+1}{x^3} = \frac{1}{-0} = -\infty$$

$$\lim_{x\to \pm\infty} \frac{3x+1}{x^3} = \lim_{x\to \pm\infty} \frac{3}{x^2} = \frac{3}{\infty}$$

Dosadíme.

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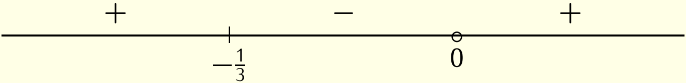
$$\boxed{y = \frac{3x+1}{x^3}} \quad D(f) = \mathbb{R} \setminus \{0\} \ ;$$



$$\lim_{x\to 0^+} \frac{3x+1}{x^3} = \frac{1}{+0} = \infty$$

$$\lim_{x\to 0^-} \frac{3x+1}{x^3} = \frac{1}{-0} = -\infty$$

$$\lim_{x\to \pm\infty} \frac{3x+1}{x^3} = \lim_{x\to \pm\infty} \frac{3}{x^2} = \frac{3}{\infty} = 0$$

Limita je vypočtena.
Funkce má vodorovnou asymptotu $y = 0$ v $\pm\infty$.

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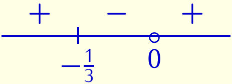
$\boxed{y = \dfrac{3x+1}{x^3}}$ $D(f) = \mathbb{R} \setminus \{0\}$;



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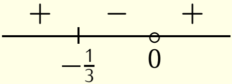
$$\boxed{y = \frac{3x+1}{x^3}}$$ $D(f) = \mathbb{R} \setminus \{0\}$;



$$y' = \frac{3x^3 - (3x+1)3x^2}{(x^3)^2}$$

Derivujeme podíl.

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

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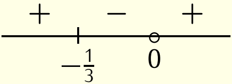
$$\boxed{y = \frac{3x+1}{x^3}}$$ $D(f) = \mathbb{R} \setminus \{0\}$;



$$y' = \frac{3x^3 - (3x+1)3x^2}{(x^3)^2} = \frac{3x^2\Big(x - (3x+1)\Big)}{x^6}$$

Vytknutím rozložíme na součin.

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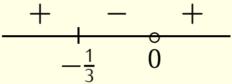
$$\boxed{y=\frac{3x+1}{x^3}}\quad D(f)=\mathbb{R}\setminus\{0\}\ ;$$



$$y'=\frac{3x^3-(3x+1)3x^2}{(x^3)^2}=\frac{3x^2\Big(x-(3x+1)\Big)}{x^6}$$

$$=3\frac{x-3x-1}{x^4}$$

- Zkrátíme.
- Roznásobíme závorku.

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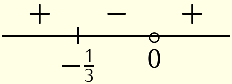
$$\boxed{y = \frac{3x+1}{x^3}}$$ $D(f) = \mathbb{R} \setminus \{0\}$;

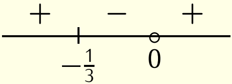


$$y' = \frac{3x^3 - (3x+1)3x^2}{(x^3)^2} = \frac{3x^2\Big(x - (3x+1)\Big)}{x^6}$$

$$= 3\frac{x - 3x - 1}{x^4} = 3\frac{-2x-1}{x^4}$$

Zjednodušíme.

$\boxed{y=\frac{3x+1}{x^3}}$ $D(f)=\mathbb{R}\setminus\{0\}$;



$$y'=\frac{3x^3-(3x+1)3x^2}{(x^3)^2}=\frac{3x^2\Big(x-(3x+1)\Big)}{x^6}$$

$$=3\frac{x-3x-1}{x^4}=3\frac{-2x-1}{x^4}=-3\frac{2x+1}{x^4}$$

Máme derivaci.

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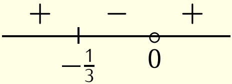
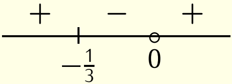
$$\boxed{y=\frac{3x+1}{x^3}}$$ $D(f)=\mathbb{R}\setminus\{0\}$;



$$y'(x)=-3\frac{2x+1}{x^4}\ ;$$

Máme derivaci.

$\boxed{y = \dfrac{3x+1}{x^3}}$ $D(f) = \mathbb{R} \setminus \{0\}$;



$y'(x) = -3\dfrac{2x+1}{x^4}$; $x_1 = -\dfrac{1}{2}$

Rovnice $y' = 0$ je ekvivalentní rovnici $2x + 1 = 0$.

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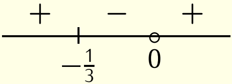
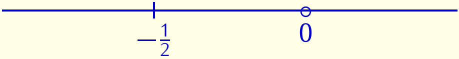
$$\boxed{y = \frac{3x+1}{x^3}}$$ $D(f) = \mathbb{R} \setminus \{0\}$;



$$y'(x) = -3\frac{2x+1}{x^4} \ ; \ \ x_1 = -\frac{1}{2}$$



Vyznačíme stacionární bod a bod nespojitosti na osu x.

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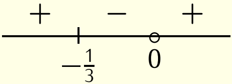
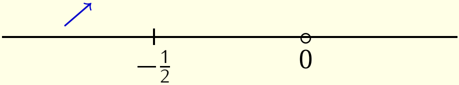
$$\boxed{y = \frac{3x+1}{x^3}}$$ $D(f) = \mathbb{R} \setminus \{0\}$;



$$y'(x) = -3\frac{2x+1}{x^4} \ ; \ \ x_1 = -\frac{1}{2}$$



$$y'(-1) = -3\frac{-2+1}{1} = 3 > 0$$

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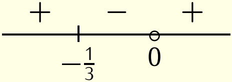
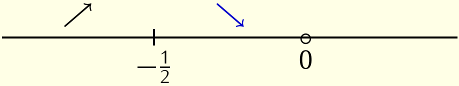
$\boxed{y = \frac{3x+1}{x^3}}$ $D(f) = \mathbb{R} \setminus \{0\}$;

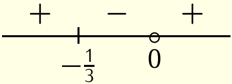


$$y'(x) = -3\frac{2x+1}{x^4} \; ; \; x_1 = -\frac{1}{2}$$

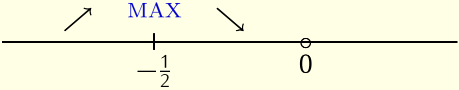


$y'(-\frac{1}{3}) < 0$, protože funkce mění znaménko z kladného na záporné.

$$\boxed{y=\frac{3x+1}{x^3}}\quad D(f)=\mathbb{R}\setminus\{0\}\ ;$$



$$y'(x)=-3\frac{2x+1}{x^4}\ ;\quad x_1=-\frac{1}{2}$$



Funkce má lokální minimum v bodě $x=-\dfrac{1}{2}$. Funkční hodnota je

$$y(-\frac{1}{2})=\frac{-\frac{3}{2}+1}{-\frac{1}{8}}=\frac{-\frac{1}{2}}{-\frac{1}{8}}=4.$$

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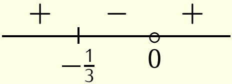
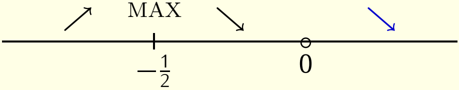
$$\boxed{y = \frac{3x+1}{x^3}}$$

$D(f) = \mathbb{R} \setminus \{0\}$;



$$y'(x) = -3\frac{2x+1}{x^4} \ ; \ x_1 = -\frac{1}{2}$$



$$y'(1) = -3\frac{3}{1} = -9 < 0$$

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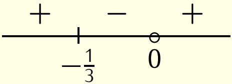
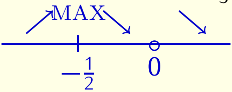
$$\boxed{y = \frac{3x+1}{x^3}}$$

$D(f) = \mathbb{R} \setminus \{0\}$;



$$y'(x) = -3\frac{2x+1}{x^4} \;;$$



$$y'' = -3\left(\frac{2x+1}{x^4}\right)'$$

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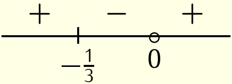
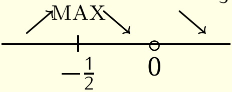
$\boxed{y=\frac{3x+1}{x^3}}$ $D(f)=\mathbb{R}\setminus\{0\}$;

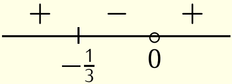


$y'(x)=-3\frac{2x+1}{x^4}$;

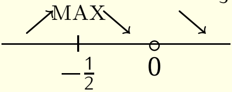


$$y''=-3\left(\frac{2x+1}{x^4}\right)'=-3\frac{2x^4-(2x+1)4x^3}{(x^4)^2}$$

$$\boxed{y=\frac{3x+1}{x^3}}\quad D(f)=\mathbb{R}\setminus\{0\}\ ;$$



$$y'(x)=-3\frac{2x+1}{x^4}\ ;$$



$$y''=-3\left(\frac{2x+1}{x^4}\right)'=-3\frac{2x^4-(2x+1)4x^3}{(x^4)^2}$$

$$=-3\frac{2x^4-8x^4-4x^3}{x^8}$$

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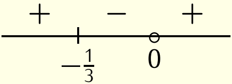
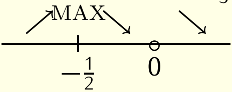
$$\boxed{y=\frac{3x+1}{x^3}}$$ $D(f)=\mathbb{R}\setminus\{0\}$;



$$y'(x)=-3\frac{2x+1}{x^4}\ ;$$



$$y''=-3\left(\frac{2x+1}{x^4}\right)'=-3\frac{2x^4-(2x+1)4x^3}{(x^4)^2}$$

$$=-3\frac{2x^4-8x^4-4x^3}{x^8}=-3\frac{-6x^4-4x^3}{x^8}$$

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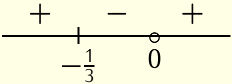
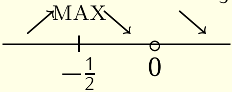
$\boxed{y=\frac{3x+1}{x^3}}$ $D(f)=\mathbb{R}\setminus\{0\}$;



$y'(x)=-3\frac{2x+1}{x^4}$;



$$y''=-3\left(\frac{2x+1}{x^4}\right)'=-3\frac{2x^4-(2x+1)4x^3}{(x^4)^2}$$

$$=-3\frac{2x^4-8x^4-4x^3}{x^8}=-3\frac{-6x^4-4x^3}{x^8}$$

$$=6\frac{3x^4+2x^3}{x^8}$$

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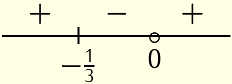
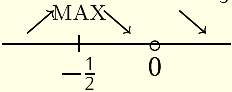
$$\boxed{y = \frac{3x+1}{x^3}}$$ $D(f) = \mathbb{R} \setminus \{0\}$;



$$y'(x) = -3\frac{2x+1}{x^4}$$;



$$y'' = -3\left(\frac{2x+1}{x^4}\right)' = -3\frac{2x^4-(2x+1)4x^3}{(x^4)^2}$$

$$= -3\frac{2x^4-8x^4-4x^3}{x^8} = -3\frac{-6x^4-4x^3}{x^8}$$

$$= 6\frac{3x^4+2x^3}{x^8} = 6\frac{(3x+2)x^3}{x^8}$$

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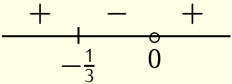
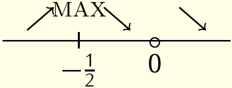
$$\boxed{y = \frac{3x+1}{x^3}}$$ $D(f) = \mathbb{R} \setminus \{0\}$;



$+ \quad - \quad +$

$-\frac{1}{3} \quad 0$

$$y'(x) = -3\frac{2x+1}{x^4}\ ;$$



$\nearrow$ MAX $\searrow \quad \searrow$

$-\frac{1}{2} \quad 0$

$$
\begin{aligned}
y'' &= -3\left(\frac{2x+1}{x^4}\right)' = -3\frac{2x^4-(2x+1)4x^3}{(x^4)^2}\\
&= -3\frac{2x^4-8x^4-4x^3}{x^8} = -3\frac{-6x^4-4x^3}{x^8}\\
&= 6\frac{3x^4+2x^3}{x^8} = 6\frac{(3x+2)x^3}{x^8}\\
&= 6\frac{3x+2}{x^5}
\end{aligned}
$$

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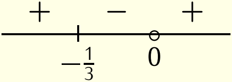
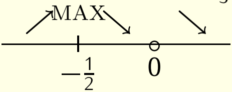
$\boxed{y = \frac{3x+1}{x^3}}$ $D(f) = \mathbb{R} \setminus \{0\}$;



$y'(x) = -3\frac{2x+1}{x^4}$;



$y'' = 6\frac{3x+2}{x^5}$;

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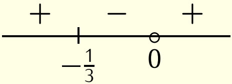
$\boxed{y = \dfrac{3x+1}{x^3}}$ $D(f) = \mathbb{R} \setminus \{0\}$;



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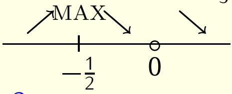

$y'' = 6\dfrac{3x+2}{x^5}$; $x_2 = -\dfrac{2}{3}$

$y'' = 0$ pro $3x+2=0$, t.j. $x = -\dfrac{2}{3}$.

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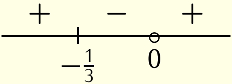
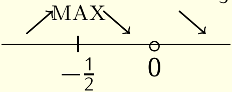
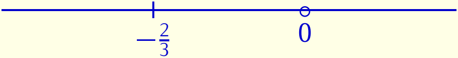
$\boxed{y = \dfrac{3x+1}{x^3}}$ $D(f) = \mathbb{R} \setminus \{0\}$;



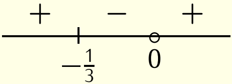
$y'(x) = -3\dfrac{2x+1}{x^4}$;



$y'' = 6\dfrac{3x+2}{x^5}$; $x_2 = -\dfrac{2}{3}$



$$\boxed{y = \frac{3x+1}{x^3}}$$ $D(f) = \mathbb{R} \setminus \{0\}$;



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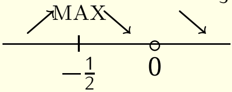
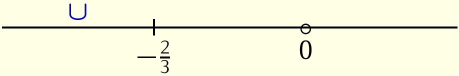


$$y'' = 6\frac{3x+2}{x^5}$$; $x_2 = -\frac{2}{3}$



$$y''(-1) = 6\frac{-1}{-1} = 6 > 0$$

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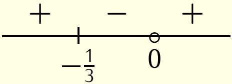
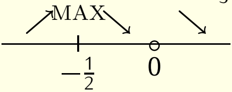
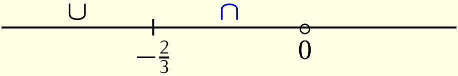
$$\boxed{y = \frac{3x+1}{x^3}}\quad D(f) = \mathbb{R} \setminus \{0\}\ ;$$



$$y'(x) = -3\frac{2x+1}{x^4}\ ;$$



$$y'' = 6\frac{3x+2}{x^5}\ ;\ x_2 = -\frac{2}{3}$$



$$y''(-\frac{1}{3}) = 6\frac{-1+2}{-\frac{1}{3^5}} < 0$$

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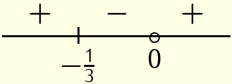
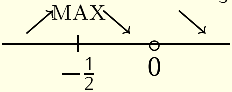
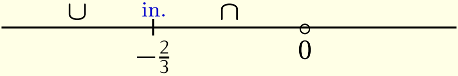
$$\boxed{y = \frac{3x+1}{x^3}}$$ $D(f) = \mathbb{R} \setminus \{0\}$;



$$y'(x) = -3\frac{2x+1}{x^4} \;;$$



$$y'' = 6\frac{3x+2}{x^5} \;;\; x_2 = -\frac{2}{3}$$



Inflexní bod $x = -\dfrac{2}{3}$. $y(-\dfrac{2}{3}) = \dfrac{-2+1}{-\frac{2^5}{3^5}} \approx 3.375$

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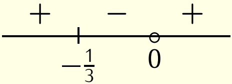
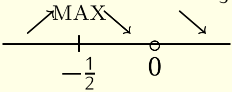
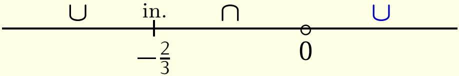
$\boxed{y=\frac{3x+1}{x^3}}$ $D(f)=\mathbb{R}\setminus\{0\}$;



$y'(x)=-3\frac{2x+1}{x^4}$;



$y''=6\frac{3x+2}{x^5}$; $x_2=-\frac{2}{3}$



$$y''(1)=6\frac{5}{1}=30>0$$

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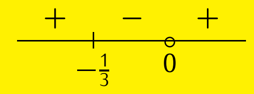

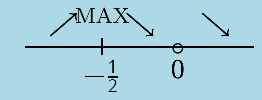

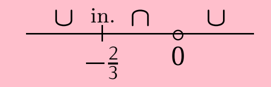

$$f(-\frac{1}{3}) = 0$$
$$f(-\frac{1}{2}) = 4$$

$$f(-\frac{2}{3}) \approx 3.4$$
$$f(\pm\infty) = 0,$$

$$f(0+) = \infty,$$
$$f(0-) = -\infty$$

Shrneme dosažené výsledky.

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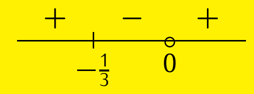

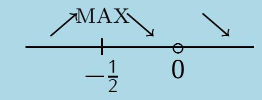

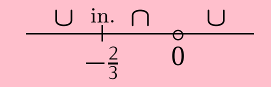

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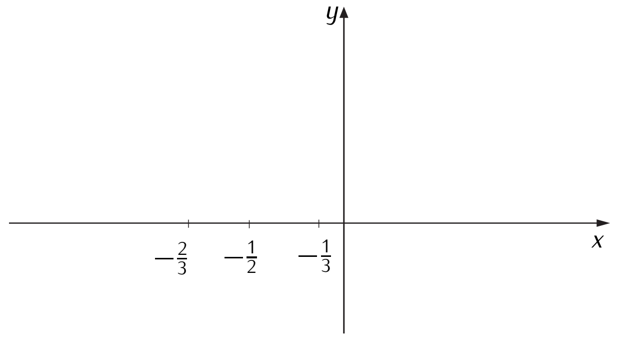

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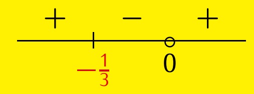

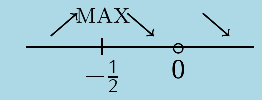

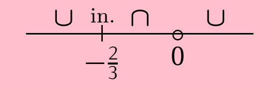

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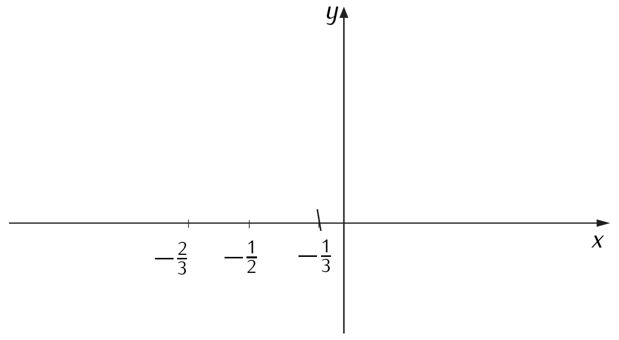

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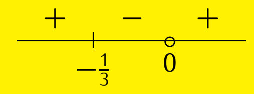

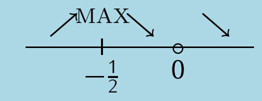

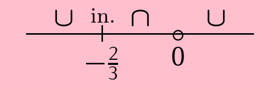

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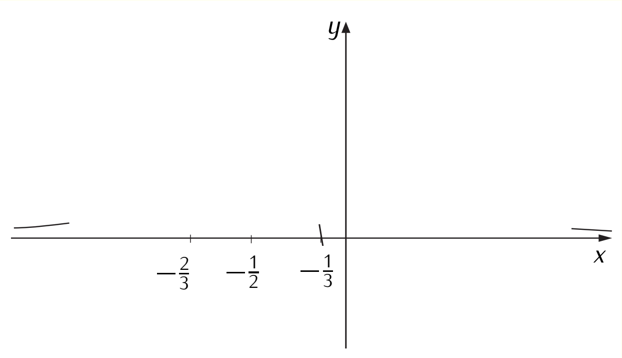

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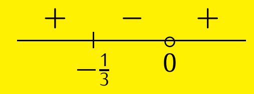

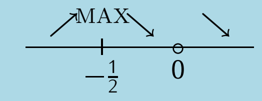

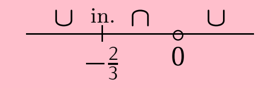

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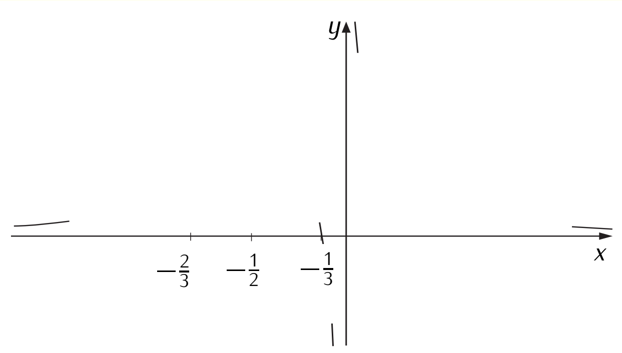

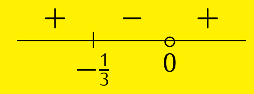

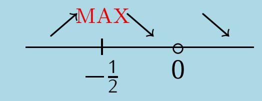

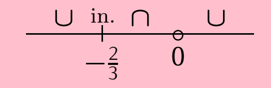

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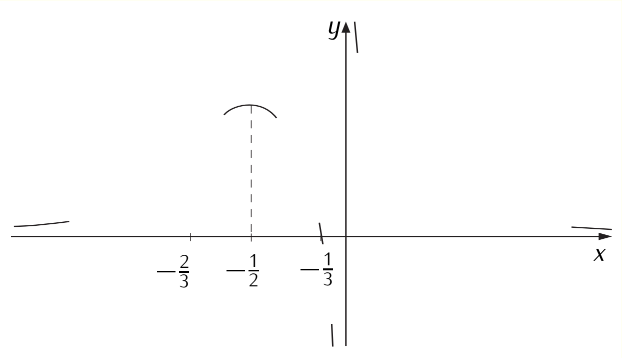

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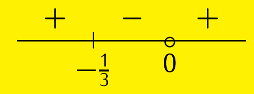

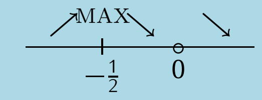

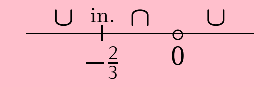

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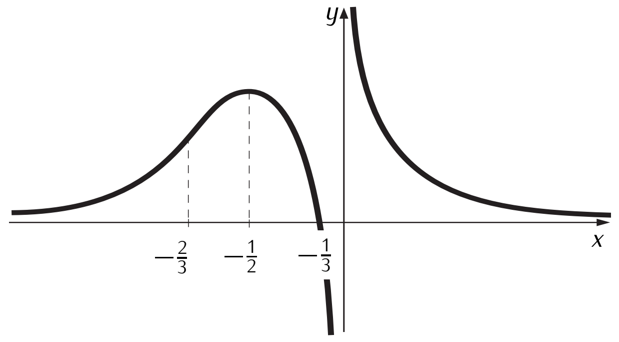

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$$\boxed{y=\frac{2(x^2-x+1)}{(x-1)^2}}$$

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$$\boxed{y = \frac{2(x^2 - x + 1)}{(x-1)^2}}$$ $D(f) = \mathbb{R} \setminus \{1\};$

Určíme definiční obor z podmínky

$$x - 1 \neq 0.$$

Platí

$$x \neq 1.$$

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$$\boxed{y=\frac{2(x^2-x+1)}{(x-1)^2}} \quad D(f)=\mathbb{R}\setminus\{1\};$$

$$y(0)=\frac{2(0-0+1)}{(0-1)^2}=2$$

- Určíme průsečík s osou y.
- Dosadíme $x=0$ a hledáme $y(0)$.

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$$\boxed{y = \frac{2(x^2 - x + 1)}{(x-1)^2}}$$ $D(f) = \mathbb{R} \setminus \{1\}$; $y(0) = 2$;

$$\frac{2(x^2 - x + 1)}{(x-1)^2} = 0$$

- Určíme průsečík s osou x.
- Dosadíme $y = 0$ a řešíme rovnici

$$\boxed{y = \frac{2(x^2 - x + 1)}{(x-1)^2}}$$ $D(f) = \mathbb{R} \setminus \{1\};\ y(0) = 2;$

$$\frac{2(x^2 - x + 1)}{(x-1)^2} = 0$$

$$x^2 - x + 1 = 0$$

Čitatel musí být nula.

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$$\boxed{y = \frac{2(x^2 - x + 1)}{(x-1)^2}}$$ $D(f) = \mathbb{R} \setminus \{1\}$; $y(0) = 2$; není průsečík s osou x

$$\frac{2(x^2 - x + 1)}{(x-1)^2} = 0$$
$$x^2 - x + 1 = 0$$

Tato kvadratická rovnice nemá řešení, protože ze vzorce

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Obdržíme záporný diskriminant.

$$D = b^2 - 4ac = 1 - 4.1.1 = -3 < 0$$

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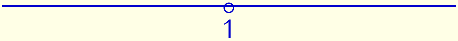
$$\boxed{y=\frac{2(x^2-x+1)}{(x-1)^2}}$$

$D(f)=\mathbb{R}\setminus\{1\}$; $y(0)=2$; není průsečík s osou x



Nakreslíme osu x a bod nespojitosti $x=1$.

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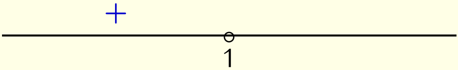
$$\boxed{y = \frac{2(x^2 - x + 1)}{(x-1)^2}}$$

$D(f) = \mathbb{R} \setminus \{1\}$; $y(0) = 2$; není průsečík s osou x



Víme, že $y(0) = 2 > 0$. Funkce je kladná na $(-\infty, 1)$.

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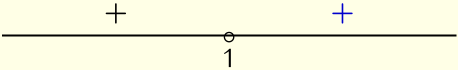
$$y = \frac{2(x^2 - x + 1)}{(x-1)^2}$$

$D(f) = \mathbb{R} \setminus \{1\}$; $y(0) = 2$; není průsečík s osou x



Vypočteme $y(2) = \dfrac{2(4-2+1)}{(2-1)^2} > 0$. Funkce je kladná na $(1, \infty)$.

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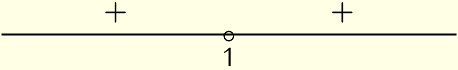
$$\boxed{y = \frac{2(x^2 - x + 1)}{(x-1)^2}}$$

$D(f) = \mathbb{R} \setminus \{1\}$; $y(0) = 2$; není průsečík s osou x



$$\lim_{x\to 1^+} \frac{2(x^2 - x + 1)}{(x-1)^2}$$

$$\lim_{x\to 1^-} \frac{2(x^2 - x + 1)}{(x-1)^2}$$

Určíme jednostranné limity v bodě nespojitosti

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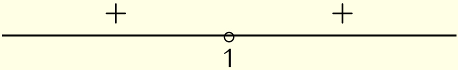
$$\boxed{y = \frac{2(x^2 - x + 1)}{(x-1)^2}}$$ $D(f) = \mathbb{R} \setminus \{1\}$; $y(0) = 2$; není průsečík s osou x



$$\lim_{x\to 1^+} \frac{2(x^2 - x + 1)}{(x-1)^2} = \frac{2}{0}$$

$$\lim_{x\to 1^-} \frac{2(x^2 - x + 1)}{(x-1)^2} = \frac{2}{0}$$

Dosadíme $x = 1$.

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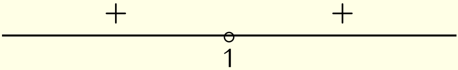
$$\boxed{y = \frac{2(x^2 - x + 1)}{(x-1)^2}}$$ $D(f) = \mathbb{R} \setminus \{1\}$; $y(0) = 2$; není průsečík s osou x



$$\lim_{x\to 1^+} \frac{2(x^2 - x + 1)}{(x-1)^2} = \frac{2}{+0} = +\infty$$

$$\lim_{x\to 1^-} \frac{2(x^2 - x + 1)}{(x-1)^2} = \frac{2}{+0} = +\infty$$

Odvodíme výsledek.

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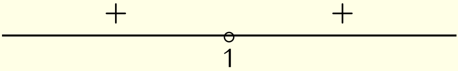
$$\boxed{y = \frac{2(x^2 - x + 1)}{(x-1)^2}}$$

$D(f) = \mathbb{R} \setminus \{1\}$; $y(0) = 2$; není průsečík s osou x



$$\lim_{x\to 1^+} \frac{2(x^2 - x + 1)}{(x-1)^2} = \frac{2}{+0} = +\infty$$

$$\lim_{x\to 1^-} \frac{2(x^2 - x + 1)}{(x-1)^2} = \frac{2}{+0} = +\infty$$

$$\lim_{x\to \pm\infty} \frac{2(x^2 - x + 1)}{(x-1)^2}$$

Určíme limity v $\pm\infty$.

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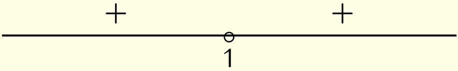
$$\boxed{y=\frac{2(x^2-x+1)}{(x-1)^2}}$$

$D(f)=\mathbb{R}\setminus\{1\}$; $y(0)=2$; není průsečík s osou x



$$\lim_{x\to 1^+}\frac{2(x^2-x+1)}{(x-1)^2}=\frac{2}{+0}=+\infty$$

$$\lim_{x\to 1^-}\frac{2(x^2-x+1)}{(x-1)^2}=\frac{2}{+0}=+\infty$$

$$\lim_{x\to \pm\infty}\frac{2(x^2-x+1)}{(x-1)^2}=\lim_{x\to\pm\infty}\frac{2x^2}{x^2}=$$

Uvažujeme jenom vedoucí členy.

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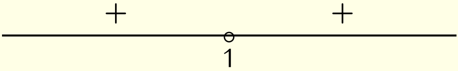
$$\boxed{y = \frac{2(x^2 - x + 1)}{(x-1)^2}}$$

$D(f) = \mathbb{R} \setminus \{1\}$; $y(0) = 2$; není průsečík s osou x



$$\lim_{x\to 1^+} \frac{2(x^2 - x + 1)}{(x-1)^2} = \frac{2}{+0} = +\infty$$

$$\lim_{x\to 1^-} \frac{2(x^2 - x + 1)}{(x-1)^2} = \frac{2}{+0} = +\infty$$

$$\lim_{x\to \pm\infty} \frac{2(x^2 - x + 1)}{(x-1)^2} = \lim_{x\to \pm\infty} \frac{2x^2}{x^2} = \lim_{x\to \pm\infty} \frac{2}{1} = 2$$

Funkce má kladnou limitu v $\pm\infty$. Vodorovná přímka $y = 2$ je asymptotou ke grafu v bodech $\pm\infty$.

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$$\boxed{y = \frac{2(x^2 - x + 1)}{(x-1)^2}}$$ $D(f) = \mathbb{R} \setminus \{1\}$; $y(0) = 2$; není průsečík s osou x

$$y' = 2\left(\frac{x^2 - x + 1}{(x-1)^2}\right)'$$

Vypočteme derivaci

$\boxed{y=\frac{2(x^2-x+1)}{(x-1)^2}}$ $D(f)=\mathbb{R}\setminus\{1\}$; $y(0)=2$; není průsečík s osou x

$$
\begin{aligned}
y' &= 2\left(\frac{x^2-x+1}{(x-1)^2}\right)'\\
&= 2\frac{(2x-1)(x-1)^2-(x^2-x+1)2(x-1)(1-0)}{((x-1)^2)^2}
\end{aligned}
$$

- Užijeme vzorec pro derivaci podílu.

$$\left(\frac{u}{v}\right)'=\frac{u'v-uv'}{v^2}.$$

- Užijeme vzorec pro derivaci složené funkce při derivování výrazu $(x-1)^2$.

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$$\boxed{y = \frac{2(x^2 - x + 1)}{(x-1)^2}}$$

$D(f) = \mathbb{R} \setminus \{1\}$; $y(0) = 2$; není průsečík s osou x

$$\begin{aligned} y' &= 2\left(\frac{x^2 - x + 1}{(x-1)^2}\right)' \\ &= 2\frac{(2x-1)(x-1)^2 - (x^2 - x + 1)2(x-1)(1-0)}{((x-1)^2)^2} \\ &= 2(x-1)\frac{(2x-1)(x-1) - (x^2 - x + 1)2}{(x-1)^4} \end{aligned}$$

Vytkneme $(x-1)$.

$$\boxed{y = \frac{2(x^2 - x + 1)}{(x-1)^2}}$$

$D(f) = \mathbb{R} \setminus \{1\}$; $y(0) = 2$; není průsečík s osou x

$$\begin{aligned} y' &= 2\left(\frac{x^2 - x + 1}{(x-1)^2}\right)' \\ &= 2\frac{(2x-1)(x-1)^2 - (x^2 - x + 1)2(x-1)(1-0)}{((x-1)^2)^2} \\ &= 2(x-1)\frac{(2x-1)(x-1) - (x^2 - x + 1)2}{(x-1)^4} \\ &= 2\frac{2x^2 - 2x - x + 1 - (2x^2 - 2x + 2)}{(x-1)^3} \end{aligned}$$

Roznásobíme závorky a zkrátíme $(x-1)$.

$$\boxed{y = \frac{2(x^2 - x + 1)}{(x-1)^2}}$$ $D(f) = \mathbb{R} \setminus \{1\}$; $y(0) = 2$; není průsečík s osou x

$$\begin{aligned}
y' &= 2\left(\frac{x^2 - x + 1}{(x-1)^2}\right)' \\
&= 2\frac{(2x-1)(x-1)^2 - (x^2 - x + 1)2(x-1)(1-0)}{((x-1)^2)^2} \\
&= 2(x-1)\frac{(2x-1)(x-1) - (x^2 - x + 1)2}{(x-1)^4} \\
&= 2\frac{2x^2 - 2x - x + 1 - (2x^2 - 2x + 2)}{(x-1)^3} \\
&= 2\frac{-x-1}{(x-1)^3}
\end{aligned}$$

Upravíme čitatel.

$$\boxed{y = \frac{2(x^2 - x + 1)}{(x-1)^2}}$$ $D(f) = \mathbb{R} \setminus \{1\}$; $y(0) = 2$; není průsečík s osou x

$$\begin{aligned} y' &= 2\left(\frac{x^2 - x + 1}{(x-1)^2}\right)' \\ &= 2\frac{(2x-1)(x-1)^2 - (x^2 - x + 1)2(x-1)(1-0)}{((x-1)^2)^2} \\ &= 2(x-1)\frac{(2x-1)(x-1) - (x^2 - x + 1)2}{(x-1)^4} \\ &= 2\frac{2x^2 - 2x - x + 1 - (2x^2 - 2x + 2)}{(x-1)^3} \\ &= 2\frac{-x-1}{(x-1)^3} = -2\frac{x+1}{(x-1)^3} \end{aligned}$$

Derivace je nalezena.

$$\boxed{y = \frac{2(x^2 - x + 1)}{(x-1)^2}}$$ $D(f) = \mathbb{R} \setminus \{1\}$; $y(0) = 2$; není průsečík s osou x

$$y' = -2\frac{x+1}{(x-1)^3}\ ;$$

$$-2\frac{x+1}{(x-1)^3} = 0$$

Řešíme rovnici $y' = 0$.

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$$\boxed{y = \frac{2(x^2 - x + 1)}{(x-1)^2}}$$ $D(f) = \mathbb{R} \setminus \{1\}$; $y(0) = 2$; není průsečík s osou x

$$y' = -2\frac{x+1}{(x-1)^3} \ ;$$

$$-2\frac{x+1}{(x-1)^3} = 0$$

$$x + 1 = 0$$

$$x = -1$$

Čitatel musí být nula. Stacionárním bodem je tedy $x = -1$.

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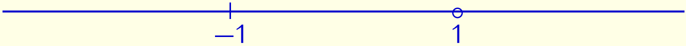
$$\boxed{y = \frac{2(x^2 - x + 1)}{(x-1)^2}}$$ $D(f) = \mathbb{R} \setminus \{1\}$; $y(0) = 2$; není průsečík s osou x

$$y' = -2\frac{x+1}{(x-1)^3}\ ;\ x_1 = -1$$



Zakreslíme stacionární bod a bod nespojitosti na reálnou osu.

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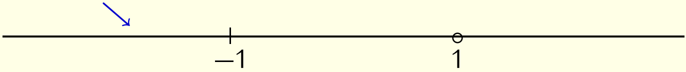
$$\boxed{y=\frac{2(x^2-x+1)}{(x-1)^2}}$$ $D(f)=\mathbb{R}\setminus\{1\}$; $y(0)=2$; není průsečík s osou x

$$y'=-2\frac{x+1}{(x-1)^3}\ ;\ x_1=-1$$



Určíme $y'(-2)$.

$$y'(-2)=-2\frac{-2+1}{(-2-1)^3}=-2\frac{\text{záp. hodnota}}{\text{záp. hodnota}}<0$$

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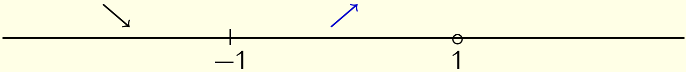
$$\boxed{y = \frac{2(x^2 - x + 1)}{(x-1)^2}}$$ $D(f) = \mathbb{R} \setminus \{1\}$; $y(0) = 2$; není průsečík s osou x

$$y' = -2\frac{x+1}{(x-1)^3} \ ; \ x_1 = -1$$



Určíme $y'(0)$.

$$y'(0) = -2\frac{0+1}{(0-1)^3} = -2\frac{\text{kladná hodnota}}{\text{záporná hodnota}} > 0$$

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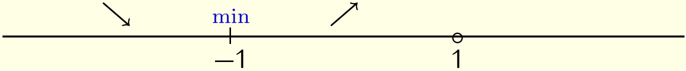
$$\boxed{y = \frac{2(x^2 - x + 1)}{(x-1)^2}}$$ $D(f) = \mathbb{R} \setminus \{1\}$; $y(0) = 2$; není průsečík s osou x

$y' = -2\frac{x+1}{(x-1)^3}$; $x_1 = -1$… lok. minimum, $y(-1) = \frac{3}{2}$



Lokální minimum je v bodě $x = -1$. Funkční hodnota je

$$y(-1) = \frac{2((-1)^2 - (-1) + 1)}{(-1-1)^2} = \frac{2.3}{4} = \frac{3}{2}.$$

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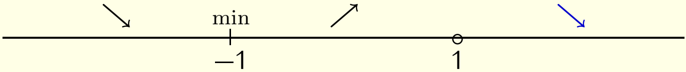
$$\boxed{y = \frac{2(x^2 - x + 1)}{(x-1)^2}}$$ $D(f) = \mathbb{R} \setminus \{1\}$; $y(0) = 2$; není průsečík s osou x

$y' = -2\frac{x+1}{(x-1)^3}$; $x_1 = -1 \ldots$ lok. minimum, $y(-1) = \frac{3}{2}$



$$y'(2) = -2\frac{2+1}{(2-1)^3} = -2\frac{3}{1} < 0$$

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$$\boxed{y = \frac{2(x^2 - x + 1)}{(x-1)^2}}$$ $D(f) = \mathbb{R} \setminus \{1\}$; $y(0) = 2$; není průsečík s osou x

$y' = -2\dfrac{x+1}{(x-1)^3}$; $x_1 = -1 \ldots$ lok. minimum, $y(-1) = \dfrac{3}{2}$

$$y'' = -2\left(\frac{x+1}{(x-1)^3}\right)'$$

Vypočteme druhou derivaci.

$$\boxed{y = \frac{2(x^2 - x + 1)}{(x-1)^2}}$$ $D(f) = \mathbb{R} \setminus \{1\}$; $y(0) = 2$; není průsečík s osou x

$y' = -2\dfrac{x+1}{(x-1)^3}$; $x_1 = -1\ldots$ lok. minimum, $y(-1) = \dfrac{3}{2}$

$$\begin{aligned} y'' &= -2\left(\frac{x+1}{(x-1)^3}\right)' \\ &= -2\frac{1(x-1)^3 - (x+1)3(x-1)^2(1-0)}{((x-1)^3)^2} \end{aligned}$$

- Použijeme pravidlo pro derivaci podílu.
- Jmenovatel budeme derivovat jako složenou funkci.

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$$\boxed{y = \frac{2(x^2 - x + 1)}{(x-1)^2}}$$ $D(f) = \mathbb{R} \setminus \{1\}$; $y(0) = 2$; není průsečík s osou x

$y' = -2\dfrac{x+1}{(x-1)^3}$; $x_1 = -1\ldots$ lok. minimum, $y(-1) = \dfrac{3}{2}$

$$\begin{aligned} y'' &= -2\left(\frac{x+1}{(x-1)^3}\right)' \\ &= -2\frac{1(x-1)^3 - (x+1)3(x-1)^2(1-0)}{((x-1)^3)^2} \\ &= -2(x-1)^2\frac{(x-1)-(x+1)3}{(x-1)^6} \end{aligned}$$

Vytkneme $(x-1)^2$ v čitateli.

$$\boxed{y = \frac{2(x^2 - x + 1)}{(x-1)^2}}$$ $D(f) = \mathbb{R} \setminus \{1\}$; $y(0) = 2$; není průsečík s osou x

$y' = -2\frac{x+1}{(x-1)^3}$; $x_1 = -1 \ldots$ lok. minimum, $y(-1) = \frac{3}{2}$

$$\begin{aligned} y'' &= -2\left(\frac{x+1}{(x-1)^3}\right)' \\ &= -2\frac{1(x-1)^3 - (x+1)3(x-1)^2(1-0)}{((x-1)^3)^2} \\ &= -2(x-1)^2\frac{(x-1)-(x+1)3}{(x-1)^6} \\ &= -2\frac{-2x-4}{(x-1)^4} \end{aligned}$$

Upravíme.

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$$\boxed{y=\frac{2(x^2-x+1)}{(x-1)^2}}$$ $D(f)=\mathbb{R}\setminus\{1\}$; $y(0)=2$; není průsečík s osou x

$y'=-2\frac{x+1}{(x-1)^3}$; $x_1=-1\ldots$ lok. minimum, $y(-1)=\frac{3}{2}$

$$\begin{aligned}
y'' &= -2\left(\frac{x+1}{(x-1)^3}\right)' \\
&= -2\frac{1(x-1)^3-(x+1)3(x-1)^2(1-0)}{((x-1)^3)^2} \\
&= -2(x-1)^2\frac{(x-1)-(x+1)3}{(x-1)^6} \\
&= -2\frac{-2x-4}{(x-1)^4} = 4\frac{x+2}{(x-1)^4}
\end{aligned}$$

Obdrželi jsme druhou derivaci.

$$\boxed{y = \frac{2(x^2 - x + 1)}{(x-1)^2}}$$ $D(f) = \mathbb{R} \setminus \{1\}$; $y(0) = 2$; není průsečík s osou x

$y' = -2\frac{x+1}{(x-1)^3}$; $x_1 = -1\ldots$ lok. minimum, $y(-1) = \frac{3}{2}$

$y'' = 4\frac{x+2}{(x-1)^4}$;

$$4\frac{x+2}{(x-1)^4} = 0$$

Řešíme $y'' = 0$.

$$\boxed{y = \frac{2(x^2 - x + 1)}{(x-1)^2}}$$ $D(f) = \mathbb{R} \setminus \{1\}$; $y(0) = 2$; není průsečík s osou x

$y' = -2\dfrac{x+1}{(x-1)^3}$; $x_1 = -1 \dots$ lok. minimum, $y(-1) = \dfrac{3}{2}$

$y'' = 4\dfrac{x+2}{(x-1)^4}$; $x_2 = -2$

$$4\frac{x+2}{(x-1)^4} = 0$$

$$x + 2 = 0$$

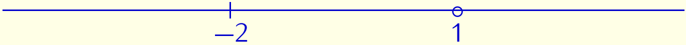
$$x = -2$$

Jediné řešení je $x = -2$.

$$\boxed{y = \frac{2(x^2 - x + 1)}{(x-1)^2}}$$ $D(f) = \mathbb{R} \setminus \{1\}$; $y(0) = 2$; není průsečík s osou x

$$y' = -2\frac{x+1}{(x-1)^3}\ ;\ x_1 = -1 \ldots \text{lok. minimum},\ y(-1) = \frac{3}{2}$$

$$y'' = 4\frac{x+2}{(x-1)^4}\ ;\ x_2 = -2$$



Budeme určovat intervaly konvexnosti a konkavity. Zakreslíme bod, kde je druhá derivace nulová a bod nespojitosti na reálnou osu.

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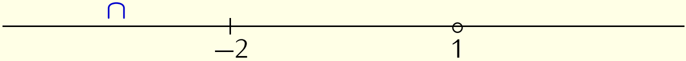
$$\boxed{y = \frac{2(x^2 - x + 1)}{(x-1)^2}}$$

$D(f) = \mathbb{R} \setminus \{1\}$; $y(0) = 2$; není průsečík s osou x

$$y' = -2\frac{x+1}{(x-1)^3}\ ;\ x_1 = -1 \ldots \text{lok. minimum, } y(-1) = \frac{3}{2}$$

$$y'' = 4\frac{x+2}{(x-1)^4}\ ;\ x_2 = -2$$



$$y''(-3) = 4\frac{-3+2}{\text{kladná hodnota}} < 0$$

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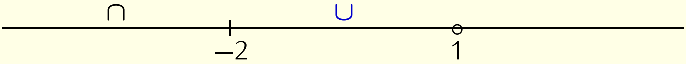
$$\boxed{y=\frac{2(x^2-x+1)}{(x-1)^2}}$$ $D(f)=\mathbb{R}\setminus\{1\}$; $y(0)=2$; není průsečík s osou x

$y'=-2\dfrac{x+1}{(x-1)^3}$; $x_1=-1\ldots$ lok. minimum, $y(-1)=\dfrac{3}{2}$

$y''=4\dfrac{x+2}{(x-1)^4}$; $x_2=-2$



$$y''(0)=4\frac{0+2}{\text{kladná hodnota}}>0$$

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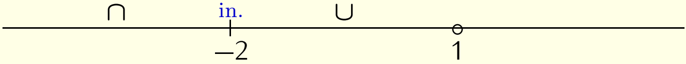
$$\boxed{y = \frac{2(x^2 - x + 1)}{(x-1)^2}}$$ $D(f) = \mathbb{R} \setminus \{1\}$; $y(0) = 2$; není průsečík s osou x

$y' = -2\frac{x+1}{(x-1)^3}$; $x_1 = -1 \ldots$ lok. minimum, $y(-1) = \frac{3}{2}$

$y'' = 4\frac{x+2}{(x-1)^4}$; $x_2 = -2$



Inflexní bod je v bodě $x = -2$. Funkční hodnota je

$$y(-2) = \frac{14}{9}.$$

(Vypočtěte si sami.)

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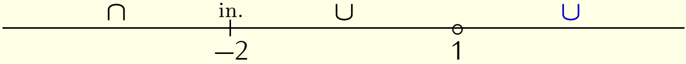
$$\boxed{y=\frac{2(x^2-x+1)}{(x-1)^2}}$$ $D(f)=\mathbb{R}\setminus\{1\}$; $y(0)=2$; není průsečík s osou x

$y'=-2\dfrac{x+1}{(x-1)^3}$; $x_1=-1\ldots$ lok. minimum, $y(-1)=\dfrac{3}{2}$

$y''=4\dfrac{x+2}{(x-1)^4}$; $x_2=-2$



$$y''(2)=4\frac{2+1}{\text{kladná hodnota}}>0$$

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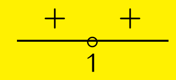

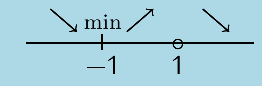

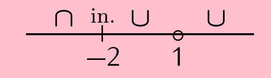

$f(0) = 2$

$f(\pm\infty) = 2$

$f(1\pm) = +\infty$

$f(-1) = \frac{3}{2}$

$f(-2) = \frac{14}{9}$

Shrneme dosavadní znalosti.

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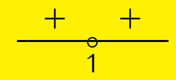

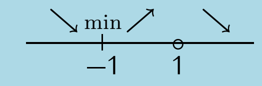

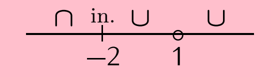

$f(0) = 2$

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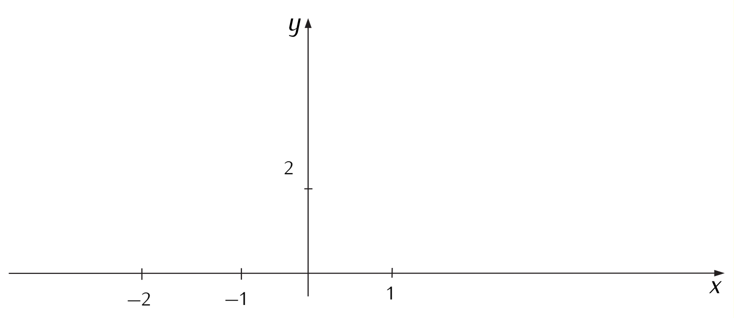

Nakreslíme souřadnou soustavu.

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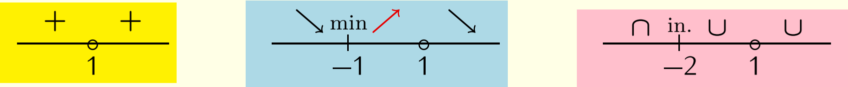

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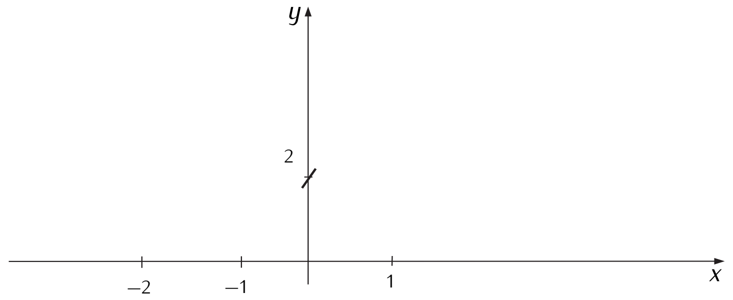

Vyznačíme průsečík s osou y. Funkce v tomto bodě roste.

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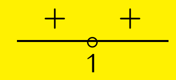

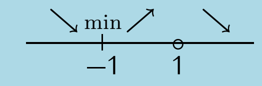

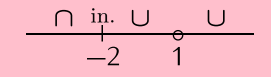

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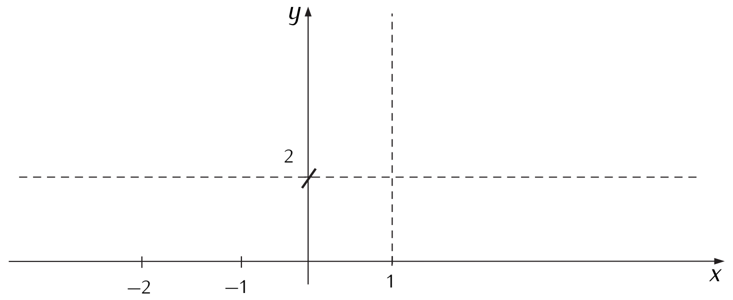

Nakreslíme asymptoty.

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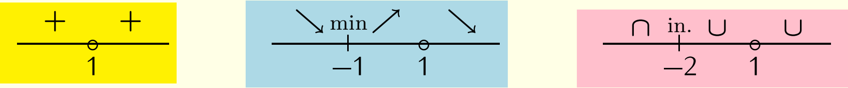

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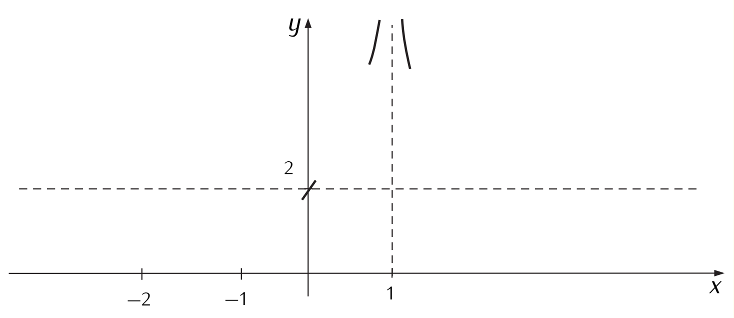

Nakreslíme funkci v okolí svislé asymptoty.

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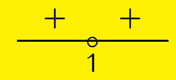

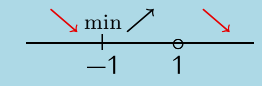

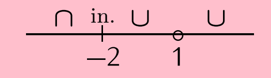

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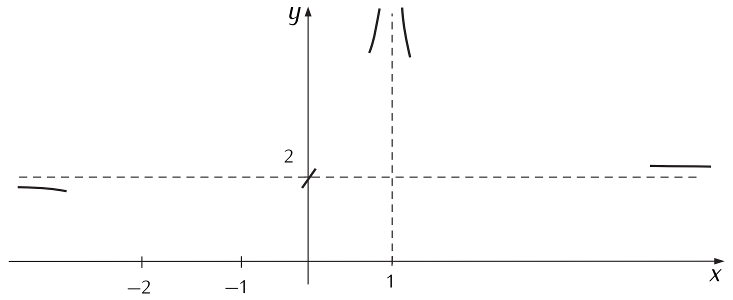

Nakreslíme funkci v okolí vodorovné asymptoty.

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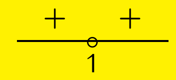

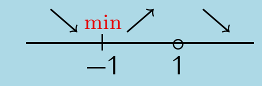

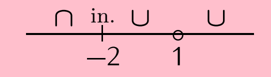

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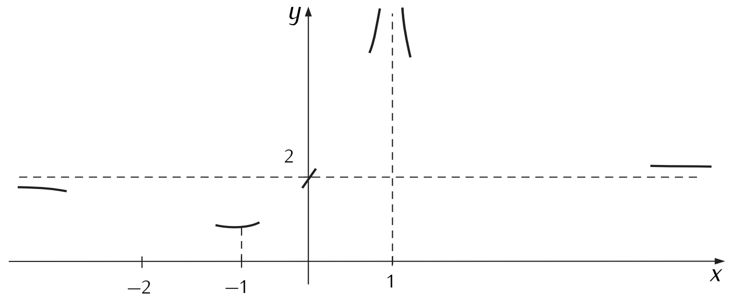

Nakreslíme lokální minimum funkce.

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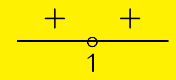
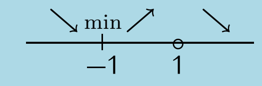
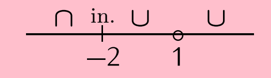






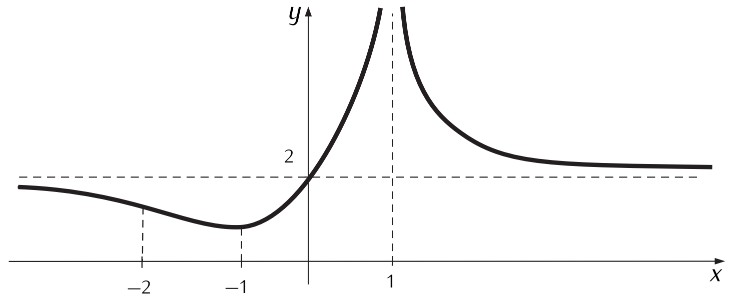
$f(0) = 2$

$f(\pm\infty) = 2$

$f(1\pm) = +\infty$

$f(-1) = \dfrac{3}{2}$

$f(-2) = \dfrac{14}{9}$



Hotovo!

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$$y = \frac{x^3}{3 - x^2}$$

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$$\boxed{y = \frac{x^3}{3 - x^2}}$$ $D(f) = \mathbb{R} \setminus \{\pm\sqrt{3}\};$

Definiční obor určíme z podmínky $3 - x^2 \neq 0$. Dostáváme dva body nespojitosti $\pm\sqrt{3}$.

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$$\boxed{y = \frac{x^3}{3 - x^2}}$$ $D(f) = \mathbb{R} \setminus \{\pm\sqrt{3}\}$; $y(0) = 0$

Průsečík s osou y má druhou souřadnici

$$y(0) = \frac{0}{3 - 0} = 0.$$

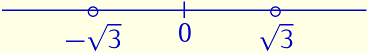
$$\boxed{y=\frac{x^3}{3-x^2}}\quad D(f)=\mathbb{R}\setminus\{\pm\sqrt{3}\};\qquad y(0)=0$$

průsečík s x: $x=0$

Řešením rovnice $\frac{x^3}{3-x^2}=0$ získáváme jediný průsečík s osou x, bod $x=0$.

$$\boxed{y = \frac{x^3}{3 - x^2}} \quad D(f) = \mathbb{R} \setminus \{\pm\sqrt{3}\}; \qquad y(0) = 0$$

průsečík s x: $x = 0$



Nulový bod a body nespojitosti vyneseme na reálnou osu.

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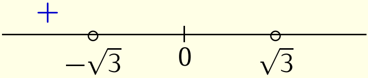
$\boxed{y = \dfrac{x^3}{3 - x^2}}$ $D(f) = \mathbb{R} \setminus \{\pm\sqrt{3}\}$; $y(0) = 0$

průsečík s x: $x = 0$



Platí

$$y(-2) = \frac{-8}{3-4} = 8 > 0$$

a graf funkce je nad osou x na intervalu $(-\infty, -\sqrt{3})$.

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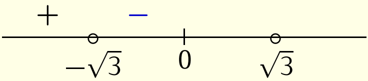
$$\boxed{y = \frac{x^3}{3-x^2}}$$ $D(f) = \mathbb{R} \setminus \{\pm\sqrt{3}\}$; $y(0) = 0$

průsečík s x: $x = 0$



Platí

$$y(-1) = \frac{-1}{3-1} = -\frac{1}{2} < 0$$

a graf funkce je pod osou x na intervalu $(-\sqrt{3}, 0)$.

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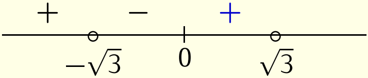
$$\boxed{y=\frac{x^3}{3-x^2}}\quad D(f)=\mathbb{R}\setminus\{\pm\sqrt{3}\};\qquad y(0)=0$$

průsečík s x: $x=0$



Platí

$$y(1)=\frac{1}{3-1}=\frac{1}{2}>0$$

a graf funkce je nad osou x na intervalu $(0,\sqrt{3})$.

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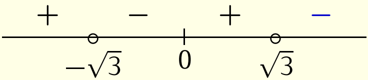
$$\boxed{y = \frac{x^3}{3-x^2}} \quad D(f) = \mathbb{R} \setminus \{\pm\sqrt{3}\}; \qquad y(0) = 0$$

průsečík s x: $x = 0$



Platí

$$y(2) = \frac{8}{3-4} = -8 < 0$$

a graf funkce je pod osou x na intervalu $(\sqrt{3}, \infty)$.

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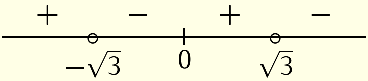
$$\boxed{y = \frac{x^3}{3 - x^2}}\quad D(f) = \mathbb{R} \setminus \{\pm\sqrt{3}\}; \qquad y(0) = 0$$

průsečík s x: $x = 0$



$$\lim_{x\to-\sqrt{3}^-} \frac{x^3}{3 - x^2}$$

$$\lim_{x\to-\sqrt{3}^+} \frac{x^3}{3 - x^2}$$

$$\lim_{x\to\sqrt{3}^-} \frac{x^3}{3 - x^2}$$

$$\lim_{x\to\sqrt{3}^+} \frac{x^3}{3 - x^2}$$

Budeme zkoumat jednostranné limity v bodech nespojitosti.

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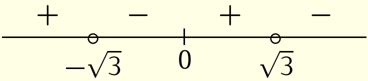
$$\boxed{y = \frac{x^3}{3-x^2}}\quad D(f) = \mathbb{R} \setminus \{\pm\sqrt{3}\}; \qquad y(0) = 0$$

průsečík s x: $x = 0$



$$\lim_{x\to-\sqrt{3}^-} \frac{x^3}{3-x^2} = \frac{-\sqrt{27}}{0} = \infty \qquad \lim_{x\to\sqrt{3}^-} \frac{x^3}{3-x^2} = \frac{\sqrt{27}}{0} = \infty$$

$$\lim_{x\to-\sqrt{3}^+} \frac{x^3}{3-x^2} = \frac{-\sqrt{27}}{0} = -\infty \qquad \lim_{x\to\sqrt{3}^+} \frac{x^3}{3-x^2} = \frac{\sqrt{27}}{0} = -\infty$$

Všechny limity jsou typu $\dfrac{\text{nenulový výraz}}{0}$ a jednostranné limity jsou nevlastní. Správné znaménko snadno zjistíme ze schematu uvedeného výše.

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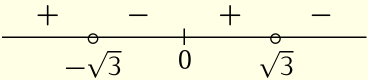
$$\boxed{y = \frac{x^3}{3-x^2}} \quad D(f) = \mathbb{R} \setminus \{\pm\sqrt{3}\}; \qquad y(0) = 0$$

průsečík s x: $x = 0$



$$\lim_{x\to-\sqrt{3}^-} \frac{x^3}{3-x^2} = \frac{-\sqrt{27}}{0} = \infty \qquad \lim_{x\to\sqrt{3}^-} \frac{x^3}{3-x^2} = \frac{\sqrt{27}}{0} = \infty$$

$$\lim_{x\to-\sqrt{3}^+} \frac{x^3}{3-x^2} = \frac{-\sqrt{27}}{0} = -\infty \qquad \lim_{x\to\sqrt{3}^+} \frac{x^3}{3-x^2} = \frac{\sqrt{27}}{0} = -\infty$$

$$\lim_{x\to\pm\infty} \frac{x^3}{3-x^2}$$

Vypočteme limity v nevlastních bodech.

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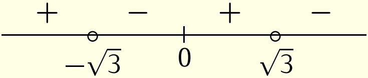
$\boxed{y = \dfrac{x^3}{3 - x^2}}$ $D(f) = \mathbb{R} \setminus \{\pm\sqrt{3}\}$; $y(0) = 0$

průsečík s x: $x = 0$


$+$ $-$ $+$ $-$

$-\sqrt{3}$ 0 $\sqrt{3}$

$$\lim_{x\to-\sqrt{3}^-} \frac{x^3}{3 - x^2} = \frac{-\sqrt{27}}{0} = \infty \qquad \lim_{x\to\sqrt{3}^-} \frac{x^3}{3 - x^2} = \frac{\sqrt{27}}{0} = \infty$$

$$\lim_{x\to-\sqrt{3}^+} \frac{x^3}{3 - x^2} = \frac{-\sqrt{27}}{0} = -\infty \qquad \lim_{x\to\sqrt{3}^+} \frac{x^3}{3 - x^2} = \frac{\sqrt{27}}{0} = -\infty$$

$$\lim_{x\to\pm\infty} \frac{x^3}{3 - x^2} = \lim_{x\to\pm\infty} \frac{x^3}{-x^2} = \lim_{x\to\pm\infty} -x = \mp\infty$$

Jedná se o podíl polynomů. V nevlastních bodech je podstatná pouze závislost na vedoucích členech polynomů v čitateli a ve jmenovateli.

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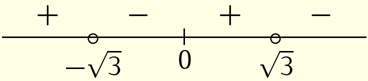
$$\boxed{y = \frac{x^3}{3-x^2}}$$ $D(f) = \mathbb{R} \setminus \{\pm\sqrt{3}\}$; $y(0) = 0$

průsečík s x: $x = 0$



$$y' = \frac{3x^2 \cdot (3 - x^2) - x^3 \cdot (0 - 2x)}{(3-x^2)^2}$$

Budeme hledat derivaci funkce. Derivujeme podíl $\frac{x^3}{3-x^2}$ podle vzorce

$$\left(\frac{u}{v}\right)' = \frac{u' \cdot v - u \cdot v'}{v^2}.$$

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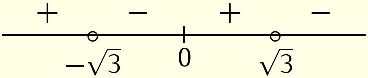
$$\boxed{y=\frac{x^3}{3-x^2}}$$ $D(f)=\mathbb{R}\setminus\{\pm\sqrt{3}\}$; $y(0)=0$

průsečík s x: $x=0$



$$y'=\frac{3x^2\cdot(3-x^2)-x^3\cdot(0-2x)}{(3-x^2)^2}$$

$$=\frac{x^2\Bigl(3(3-x^2)+2x^2\Bigr)}{(3-x^2)^2}$$

Vytkneme x^2.

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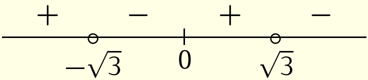
$$\boxed{y = \frac{x^3}{3-x^2}} \quad D(f) = \mathbb{R} \setminus \{\pm\sqrt{3}\}; \qquad y(0) = 0$$

průsečík s x: $x = 0$



$$y' = \frac{3x^2 \cdot (3-x^2) - x^3 \cdot (0-2x)}{(3-x^2)^2}$$

$$= \frac{x^2\Big(3(3-x^2) + 2x^2\Big)}{(3-x^2)^2}$$

$$= \frac{x^2\Big(9-x^2\Big)}{(3-x^2)^2}$$

Upravíme závorku.

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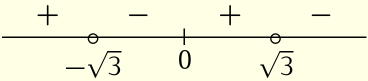
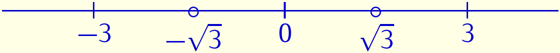
$$\boxed{y = \frac{x^3}{3-x^2}} \quad D(f) = \mathbb{R} \setminus \{\pm\sqrt{3}\}; \qquad y(0) = 0$$

průsečík s x: $x = 0$



$$y' = \frac{x^2\left(9-x^2\right)}{(3-x^2)^2} \; ;$$



Řešením rovnice $x^2(9-x^2)=0$ jsou body $x=0$ a $x=\pm 3$. Tyto stacionární body vyneseme spolu s body nespojitosti na reálnou osu.

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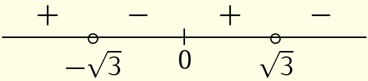
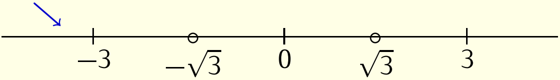
$$\boxed{y = \frac{x^3}{3 - x^2}}\quad D(f) = \mathbb{R} \setminus \{\pm\sqrt{3}\}; \qquad y(0) = 0$$

průsečík s x: $x = 0$



$$y' = \frac{x^2\left(9 - x^2\right)}{(3 - x^2)^2}\ ;$$



Červeně označené výrazy v derivaci jsou kladné a neovlivní výsledné znaménko derivace. Stačí tedy zjišťovat znaménko výrazu $(9 - x^2)$. Pro $x = -4$ platí

$$9 - x^2 = 9 - (-4)^2 < 0.$$

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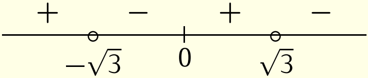
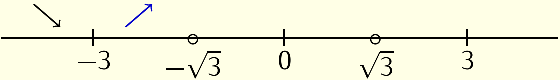
$$\boxed{y=\frac{x^3}{3-x^2}}\qquad D(f)=\mathbb{R}\setminus\{\pm\sqrt{3}\};\qquad y(0)=0$$

průsečík s x: $x=0$



$$y'=\frac{x^2\left(9-x^2\right)}{(3-x^2)^2}\ ;$$



Pro $x=-2$ platí

$$9-x^2=9-(-2)^2>0.$$

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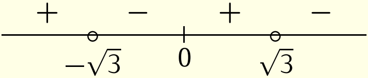
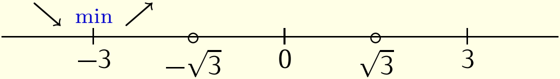
$$\boxed{y = \frac{x^3}{3 - x^2}}\quad D(f) = \mathbb{R} \setminus \{\pm\sqrt{3}\}; \qquad y(0) = 0$$

průsečík s x: $x = 0$



$$y' = \frac{{\color{red}x^2}\left(9 - x^2\right)}{{\color{red}(3 - x^2)^2}}\ ;$$



V bodě $x = -3$ je lokální minimum. Funkční hodnota je

$$y(-3) = \frac{-27}{3 - 9} = \frac{-27}{-6} = \frac{9}{2}$$

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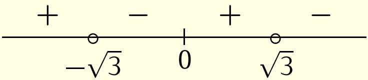
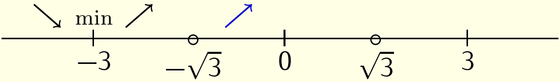
$$\boxed{y = \frac{x^3}{3-x^2}}$$ $D(f) = \mathbb{R} \setminus \{\pm\sqrt{3}\}$; $y(0) = 0$

průsečík s x: $x = 0$



$$y' = \frac{x^2\left(9-x^2\right)}{(3-x^2)^2};$$



Pro $x = -1$ platí

$$9 - x^2 = 9 - (-1)^2 > 0.$$

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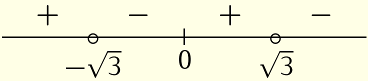
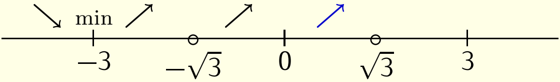
$$\boxed{y = \frac{x^3}{3 - x^2}}$$ $D(f) = \mathbb{R} \setminus \{\pm\sqrt{3}\}$; $y(0) = 0$

průsečík s x: $x = 0$



$$y' = \frac{x^2\left(9 - x^2\right)}{(3 - x^2)^2};$$



Pro $x = 1$ platí

$$9 - x^2 = 9 - 1^2 > 0.$$

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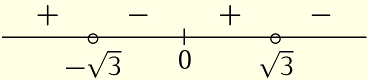
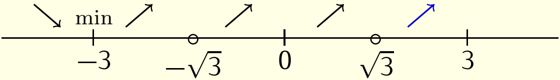
$\boxed{y = \frac{x^3}{3-x^2}}$ $D(f) = \mathbb{R} \setminus \{\pm\sqrt{3}\};$ $y(0) = 0$

průsečík s x: $x = 0$



$$y' = \frac{x^2\left(9 - x^2\right)}{(3-x^2)^2} \;;$$



Pro $x = 2$ platí

$$9 - x^2 = 9 - 2^2 > 0.$$

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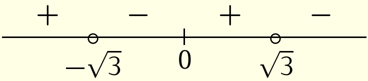
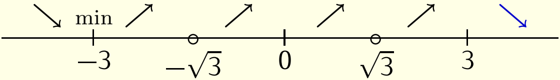
$$\boxed{y=\frac{x^3}{3-x^2}}\qquad D(f)=\mathbb{R}\setminus\{\pm\sqrt{3}\};\qquad y(0)=0$$

průsečík s x: $x=0$



$$y'=\frac{x^2\left(9-x^2\right)}{(3-x^2)^2}\ ;$$



Pro $x=4$ platí

$$9-x^2=9-4^2<0.$$

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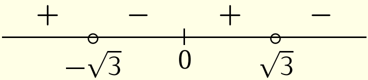
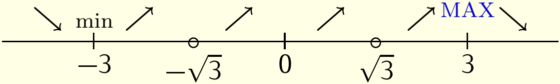
$$\boxed{y=\frac{x^3}{3-x^2}}\quad D(f)=\mathbb{R}\setminus\{\pm\sqrt{3}\};\qquad y(0)=0$$

průsečík s x: $x=0$



$$y'=\frac{x^2\left(9-x^2\right)}{(3-x^2)^2}\ ;$$



V bodě $x=3$ je lokální maximum. Funkční hodnota je

$$y(3)=\frac{27}{3-9}=\frac{27}{-6}=-\frac{9}{2}$$

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$$\boxed{y=\frac{x^3}{3-x^2}}\quad D(f)=\mathbb{R}\setminus\{\pm\sqrt{3}\};\qquad y(0)=0$$

$$y''=\frac{(18x-4x^3)\cdot(3-x^2)^2-(9x^2-x^4)\cdot 2(3-x^2)(-2x)}{\left((3-x^2)^2\right)^2}$$

Derivujeme funkci

$$\frac{x^2(9-x^2)}{(3-x^2)^2}=\frac{9x^2-x^4}{(3-x^2)^2}$$

podle vzorce

$$\left(\frac{u}{v}\right)'=\frac{u'\cdot v-u\cdot v'}{v^2}.$$

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$$\boxed{y=\frac{x^3}{3-x^2}}\quad D(f)=\mathbb{R}\setminus\{\pm\sqrt{3}\};\qquad y(0)=0$$

$$y''=\frac{(18x-4x^3)\cdot(3-x^2)^2-(9x^2-x^4)\cdot 2(3-x^2)(-2x)}{\Big((3-x^2)^2\Big)^2}$$

$$=\frac{2x(3-x^2)\cdot\Big[(9-2x^2)(3-x^2)+(9x-x^3)(2x)\Big]}{(3-x^2)^4}$$

- Protože jsme ve jmenovateli neroznásobovali, ale derivovali jako složenou funkci, nezbavili jsme se možnosti vytknout.
- Nyní tedy vytkneme členy, které se v čitateli opakují.

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$\boxed{y = \frac{x^3}{3 - x^2}}$ $D(f) = \mathbb{R} \setminus \{\pm\sqrt{3}\}$; $y(0) = 0$

$$y'' = \frac{(18x - 4x^3) \cdot (3 - x^2)^2 - (9x^2 - x^4) \cdot 2(3 - x^2)(-2x)}{\left((3 - x^2)^2\right)^2}$$

$$= \frac{2x(3 - x^2) \cdot \left[(9 - 2x^2)(3 - x^2) + (9x - x^3)(2x)\right]}{(3 - x^2)^4}$$

$$= \frac{2x \cdot \left[27 - 9x^2 - 6x^2 + 2x^4 + 18x^2 - 2x^4\right]}{(3 - x^2)^3}$$

Zkrátíme a roznásobíme závorky.

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$$\boxed{y=\frac{x^3}{3-x^2}}\quad D(f)=\mathbb{R}\setminus\{\pm\sqrt{3}\};\qquad y(0)=0$$

$$y''=\frac{(18x-4x^3)\cdot(3-x^2)^2-(9x^2-x^4)\cdot 2(3-x^2)(-2x)}{\Big((3-x^2)^2\Big)^2}$$

$$=\frac{2x(3-x^2)\cdot\Big[(9-2x^2)(3-x^2)+(9x-x^3)(2x)\Big]}{(3-x^2)^4}$$

$$=\frac{2x\cdot\Big[27-9x^2-6x^2+2x^4+18x^2-2x^4\Big]}{(3-x^2)^3}$$

$$=\frac{2x\cdot\Big[27+3x^2\Big]}{(3-x^2)^3}$$

Výrazy v hranaté závorce se sečtou resp. odečtou.

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$$\boxed{y = \frac{x^3}{3-x^2}} \quad D(f) = \mathbb{R} \setminus \{\pm\sqrt{3}\}; \qquad y(0) = 0$$

$$y'' = \frac{2x \cdot \left[27 + 3x^2\right]}{(3-x^2)^3} \; ; \; x = 0$$

Druhá derivace je vypočtena. Nyní hledáme řešení rovnice $y'' = 0$. Protože výraz $(27 + 3x^2)$ je stále kladný, je jediným řešením této rovnice bod $x = 0$.

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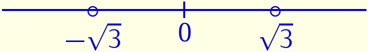
$$\boxed{y=\frac{x^3}{3-x^2}}\quad D(f)=\mathbb{R}\setminus\{\pm\sqrt{3}\};\qquad y(0)=0$$

$$y''=\frac{2x\cdot\left[27+3x^2\right]}{(3-x^2)^3}\ ;\ x=0$$



Na reálnou osu vyneseme bod $x=0$ ($y''(0)=0$) a body, kde je druhá derivace nespojitá.

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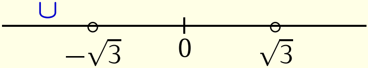
$$\boxed{y = \frac{x^3}{3 - x^2}} \quad D(f) = \mathbb{R} \setminus \{\pm\sqrt{3}\}; \qquad y(0) = 0$$

$$y'' = \frac{2x \cdot \left[27 + 3x^2\right]}{(3 - x^2)^3} \text{ ; } x = 0$$



Platí

$$y''(-2) = \frac{2 \cdot (-2) \cdot [\text{kladný výraz}]}{(3 - (-2)^2)^3} = \frac{\text{záporný výraz}}{\text{záporný výraz}} > 0$$

a funkce je konvexní na intervalu obsahujícím číslo -2.

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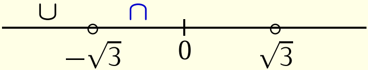
$$\boxed{y = \frac{x^3}{3 - x^2}} \quad D(f) = \mathbb{R} \setminus \{\pm\sqrt{3}\}; \qquad y(0) = 0$$

$$y'' = \frac{2x \cdot \left[27 + 3x^2\right]}{(3 - x^2)^3} \; ; \; x = 0$$



Platí

$$y''(-1) = \frac{2 \cdot (-1) \cdot [\text{kladný výraz}]}{(3 - (-1)^2)^3} = \frac{\text{záporný výraz}}{\text{kladný výraz}} < 0$$

a funkce je konkávní na intervalu obsahujícím číslo −1.

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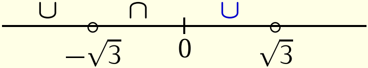
$$\boxed{y=\frac{x^3}{3-x^2}}\quad D(f)=\mathbb{R}\setminus\{\pm\sqrt{3}\};\qquad y(0)=0$$

$$y''=\frac{2x\cdot\left[27+3x^2\right]}{(3-x^2)^3}\ ;\ x=0$$



Platí

$$y''(1)=\frac{2\cdot 1\cdot[\text{kladný výraz}]}{(3-1^2)^3}=\frac{\text{kladný výraz}}{\text{kladný výraz}}>0$$

a funkce je konvexní na intervalu obsahujícím číslo 1.

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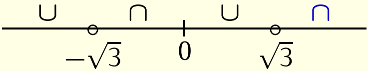
$$\boxed{y=\frac{x^3}{3-x^2}} \quad D(f)=\mathbb{R}\setminus\{\pm\sqrt{3}\}; \qquad y(0)=0$$

$$y''=\frac{2x\cdot\left[27+3x^2\right]}{(3-x^2)^3}\ ;\ x=0$$



Platí

$$y''(2)=\frac{2\cdot 2\cdot[\text{kladný výraz}]}{(3-2^2)^3}=\frac{\text{kladný výraz}}{\text{záporný výraz}}<0$$

a funkce je konkávní na intervalu obsahujícím číslo 1.

$$\boxed{y = \frac{x^3}{3-x^2}} \quad D(f) = \mathbb{R} \setminus \{\pm\sqrt{3}\}; \qquad y(0) = 0$$

Dělením se zbytkem zjistíme, že platí

$$\frac{x^3}{3-x^2} = -x + \frac{3x}{3-x^2}$$

První část je přímka, druhá část se blíží k nule pro x blížící se do plus nebo minus nekonečna.

Funkce má proto v nevlastních bodech asymptotu $y = -x$.

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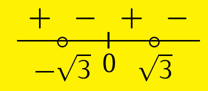

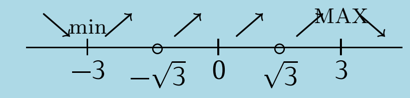

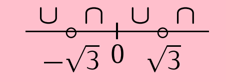

$f(0)=0;$ $f(\pm 3)=\mp\frac{9}{2}$

$f(\pm\infty)=\mp\infty;$ $f(-\sqrt{3}\pm)=\mp\infty;$ $f(\sqrt{3}\pm)=\mp\infty$

Shrneme nejdůležitější výsledky.

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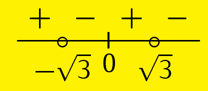
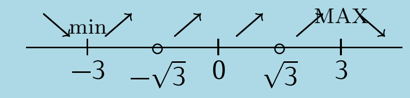
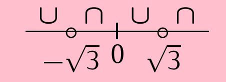
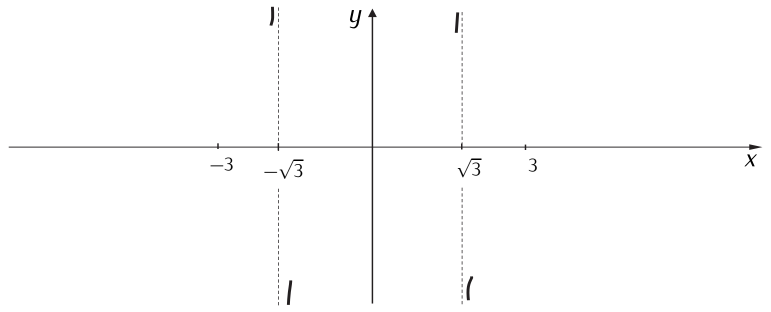






$f(0)=0;$ $f(\pm 3)=\mp\frac{9}{2}$

$f(\pm\infty)=\mp\infty;$ $f(-\sqrt{3}\pm)=\mp\infty;$ $f(\sqrt{3}\pm)=\mp\infty$



Zakreslíme svislé asymptoty a funkci v okolí těchto asymptot.

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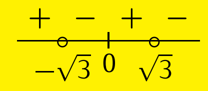
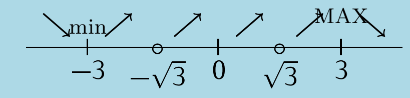


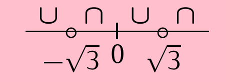
$+ \quad - \quad + \quad -$

$-\sqrt{3} \quad 0 \quad \sqrt{3}$



min MAX

$-3 \quad -\sqrt{3} \quad 0 \quad \sqrt{3} \quad 3$

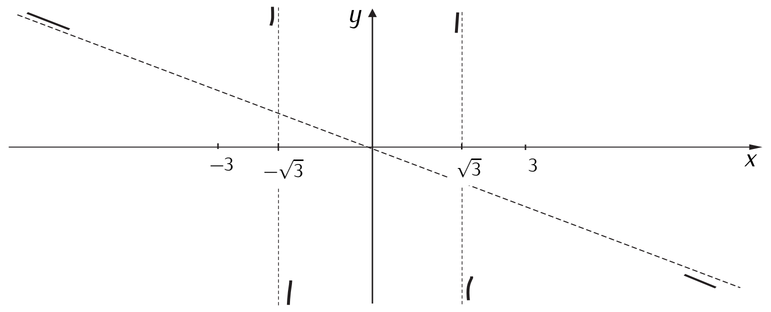


$\cup \quad \cap \quad \cup \quad \cap$

$-\sqrt{3} \quad 0 \quad \sqrt{3}$

$f(0) = 0;$ $\qquad f(\pm 3) = \mp\frac{9}{2}$

$f(\pm\infty) = \mp\infty;$ $\qquad f(-\sqrt{3}\pm) = \mp\infty;$ $\qquad f(\sqrt{3}\pm) = \mp\infty$



Podobně zakreslíme šikmou asymptotu a funkci v okolí této asymptoty. Dáváme pozor na konkavitu/konvexitu.

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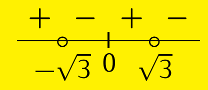
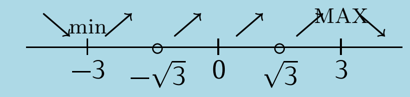
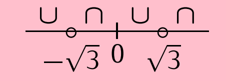
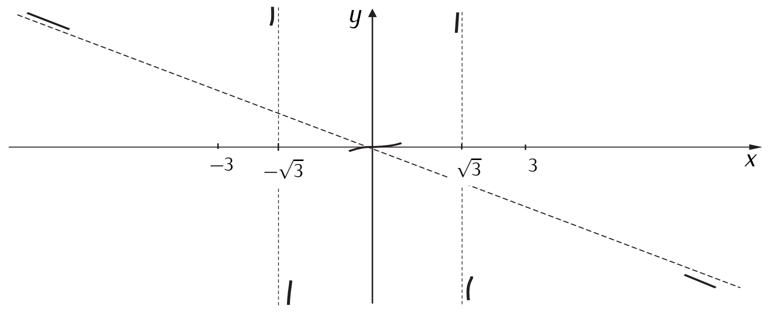






$f(0)=0;$ $f(\pm 3)=\mp\dfrac{9}{2}$

$f(\pm\infty)=\mp\infty;$ $f(-\sqrt{3}\pm)=\mp\infty;$ $f(\sqrt{3}\pm)=\mp\infty$



Zakreslíme funkci v okolí stacionárního bodu, který není lokálním extrémem.

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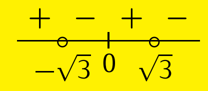
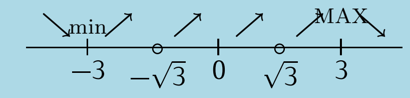
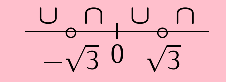
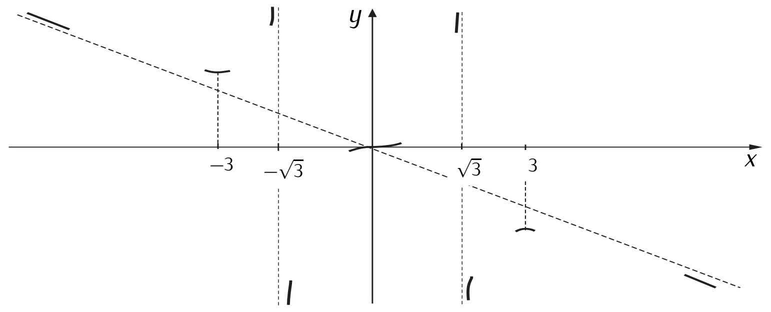




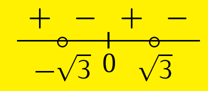


$f(0) = 0;$ $\qquad f(\pm 3) = \mp \frac{9}{2}$

$f(\pm\infty) = \mp\infty;$ $\qquad f(-\sqrt{3}\pm) = \mp\infty;$ $\qquad f(\sqrt{3}\pm) = \mp\infty$

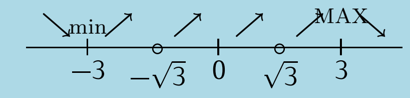


Zakreslíme lokální extrémy.



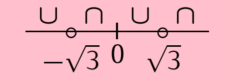
$+ \quad - \quad + \quad -$

$-\sqrt{3} \quad 0 \quad \sqrt{3}$



↘ min ↗ ↗ ↗ MAX ↘

$-3 \quad -\sqrt{3} \quad 0 \quad \sqrt{3} \quad 3$

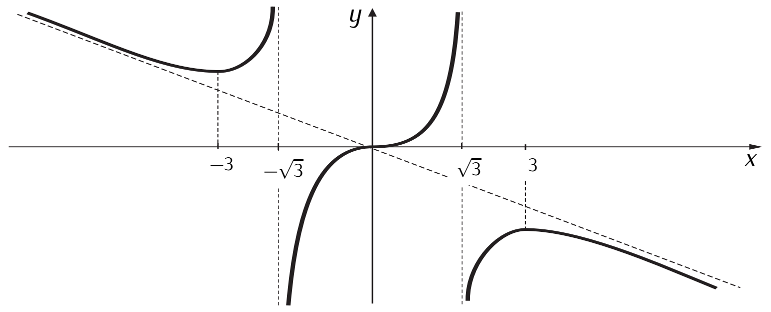


$\cup \quad \cap \quad \cup \quad \cap$

$-\sqrt{3} \quad 0 \quad \sqrt{3}$

$f(0) = 0;$ $\qquad f(\pm 3) = \mp\frac{9}{2}$

$f(\pm\infty) = \mp\infty;$ $\qquad f(-\sqrt{3}\pm) = \mp\infty;$ $\qquad f(\sqrt{3}\pm) = \mp\infty$



Dokreslíme celý graf.

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$$\boxed{y = \frac{x^2+1}{x^2-1}}$$

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$$\boxed{y = \frac{x^2+1}{x^2-1}}$$ $D(f) = \mathbb{R} \setminus \{-1, 1\};$

Určíme definiční obor – ve jmenovateli nesmí být nula. Řešením rovnice

$$x^2 - 1 = 0$$

je $x = \pm 1$. Tyto body je nutno vyloučit z definičního oboru a jedná se o body nespojitosti.

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$$\boxed{y = \frac{x^2+1}{x^2-1}}$$ $D(f) = \mathbb{R} \setminus \{-1, 1\}$; průsečík s osou y: $[0, -1]$;

Dosazením $x = 0$ určíme $y(0) = \dfrac{0+1}{0-1} = -1$, což je průsečík s osou y.

$\boxed{y = \dfrac{x^2+1}{x^2-1}}$ $D(f) = \mathbb{R} \setminus \{-1, 1\}$; průsečík s osou y: $[0, -1]$;

není průsečík s osou x

Rovnice

$$x^2 + 1 = 0$$

nemá v oboru reálných čísel řešení a funkce tedy není nikdy rovna nule. Graf nemá průsečík s osou x.

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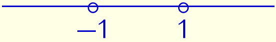
$\boxed{y = \frac{x^2+1}{x^2-1}}$ $D(f) = \mathbb{R} \setminus \{-1, 1\}$; průsečík s osou y: $[0, -1]$;

není průsečík s osou x



Znaménko funkce se může změnit nanejvýš v bodě nespojitosti (protože není průsečík s osou x). Vyneseme tedy body nespojitosti na reálnou osu.

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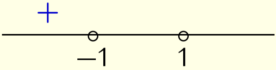
$$\boxed{y = \frac{x^2+1}{x^2-1}}$$ $D(f) = \mathbb{R} \setminus \{-1, 1\}$; průsečík s osou y: $[0, -1]$;

není průsečík s osou x



Dosazením $x = -2$ zjistíme, že $y(-2) = \frac{(-2)^2+1}{(-2)^2-1} = \frac{5}{3} > 0$ a funkce je kladná na intervalu obsahujícím číslo -2.

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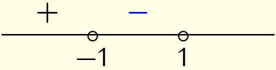
$$\boxed{y = \frac{x^2+1}{x^2-1}}$$

$D(f) = \mathbb{R} \setminus \{-1, 1\}$; průsečík s osou y: $[0, -1]$;

není průsečík s osou x



Dosazením $x = 0$ jsme již dříve zjistili (když jsme počítali průsečík s osou y), že $y(0) = -1$ a funkce je záporná na intervalu obsahujícím číslo 0.

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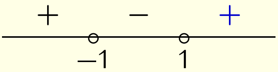
$\boxed{y = \frac{x^2+1}{x^2-1}}$ $D(f) = \mathbb{R} \setminus \{-1, 1\}$; průsečík s osou y: $[0, -1]$;

není průsečík s osou x



Dosazením $x = 2$ zjistíme, že $y(2) = \frac{(2)^2+1}{(2)^2-1} = \frac{5}{3} > 0$ a funkce je kladná na intervalu obsahujícím číslo 2.

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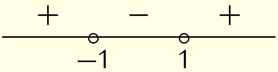
$$\boxed{y=\frac{x^2+1}{x^2-1}}$$

$D(f)=\mathbb{R}\setminus\{-1,1\}$; průsečík s osou y: $[0,-1]$;

není průsečík s osou x



$$\lim_{x\to -1^-}\frac{x^2+1}{x^2-1}=\frac{2}{0}$$
$$\lim_{x\to -1^+}\frac{x^2+1}{x^2-1}=\frac{2}{0}$$

$$\lim_{x\to 1^-}\frac{x^2+1}{x^2-1}=\frac{2}{0}$$
$$\lim_{x\to 1^+}\frac{x^2+1}{x^2-1}=\frac{2}{0}$$

Určíme jednostranné limity v bodech nespojitosti. Všechny jednostranné limity jsou typu $\frac{2}{0}$ a výsledkem budou nevlastní limity, tj. “nekonečno, opatřené správným znaménkem”.

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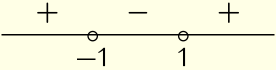
$$\boxed{y=\frac{x^2+1}{x^2-1}}$$

$D(f)=\mathbb{R}\setminus\{-1,1\}$; průsečík s osou y: $[0,-1]$;

není průsečík s osou x



$$\lim_{x\to-1^-}\frac{x^2+1}{x^2-1}=\frac{2}{0}=+\infty \qquad \lim_{x\to1^-}\frac{x^2+1}{x^2-1}=\frac{2}{0}=-\infty$$

$$\lim_{x\to-1^+}\frac{x^2+1}{x^2-1}=\frac{2}{0}=-\infty \qquad \lim_{x\to1^+}\frac{x^2+1}{x^2-1}=\frac{2}{0}=+\infty$$

Podle znamének funkce na jednotlivých podintervalech snadno odvodíme správné výsledky.

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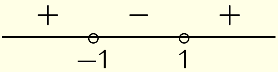
$$\boxed{y = \frac{x^2+1}{x^2-1}}$$ $D(f) = \mathbb{R} \setminus \{-1, 1\}$; průsečík s osou y: $[0, -1]$;

není průsečík s osou x



$$\lim_{x\to -1^-} \frac{x^2+1}{x^2-1} = \frac{2}{0} = +\infty \qquad \lim_{x\to 1^-} \frac{x^2+1}{x^2-1} = \frac{2}{0} = -\infty$$

$$\lim_{x\to -1^+} \frac{x^2+1}{x^2-1} = \frac{2}{0} = -\infty \qquad \lim_{x\to 1^+} \frac{x^2+1}{x^2-1} = \frac{2}{0} = +\infty$$

$$\lim_{x\to\infty} \frac{x^2+1}{x^2-1} = \lim_{x\to\infty} \frac{x^2}{x^2} = 1$$

Určíme limity v nevlastních bodech. Protože se jedná o racionální funkci, jsou pro limitu v nevlastním bodě rozhodující pouze vedoucí členy čitatele a jmenovatele.

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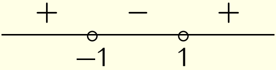
$$\boxed{y = \frac{x^2+1}{x^2-1}}$$ $D(f) = \mathbb{R} \setminus \{-1, 1\}$; průsečík s osou y: $[0, -1]$;

není průsečík s osou x



$$y' = \frac{(x^2+1)' \cdot (x^2-1) - (x^2+1) \cdot (x^2-1)'}{(x^2-1)^2}$$

Derivujeme podíl

$$y = \frac{x^2+1}{x^2-1}$$

podle vzorce

$$\left(\frac{u}{v}\right)' = \frac{u' \cdot v - u \cdot v'}{v^2}.$$

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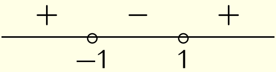
$\boxed{y = \frac{x^2+1}{x^2-1}}$ $D(f) = \mathbb{R} \setminus \{-1, 1\}$; průsečík s osou y: $[0, -1]$;

není průsečík s osou x



$$y' = \frac{(x^2+1)' \cdot (x^2-1) - (x^2+1) \cdot (x^2-1)'}{(x^2-1)^2}$$

$$= \frac{2x \cdot (x^2-1) - (x^2+1) \cdot 2x}{(x^2-1)^2}$$

Dopočítáme derivace.

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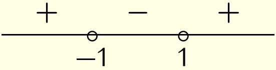
$\boxed{y = \frac{x^2+1}{x^2-1}}$ $D(f) = \mathbb{R} \setminus \{-1, 1\}$; průsečík s osou y: $[0, -1]$;

není průsečík s osou x



$$y' = \frac{(x^2+1)' \cdot (x^2-1) - (x^2+1) \cdot (x^2-1)'}{(x^2-1)^2}$$

$$= \frac{2x \cdot (x^2-1) - (x^2+1) \cdot 2x}{(x^2-1)^2}$$

$$= \frac{2x \cdot \Big(x^2 - 1 - (x^2+1)\Big)}{(x^2-1)^2}$$

Vytkneme výraz $2x$ v čitateli.

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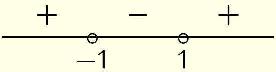
$$\boxed{y = \frac{x^2+1}{x^2-1}}$$ $D(f) = \mathbb{R} \setminus \{-1, 1\}$; průsečík s osou y: $[0, -1]$;

není průsečík s osou x



$$y' = \frac{(x^2+1)' \cdot (x^2-1) - (x^2+1) \cdot (x^2-1)'}{(x^2-1)^2}$$

$$= \frac{2x \cdot (x^2-1) - (x^2+1) \cdot 2x}{(x^2-1)^2}$$

$$= \frac{2x \cdot \left(x^2 - 1 - (x^2+1)\right)}{(x^2-1)^2}$$

$$= \frac{2x\left(-2\right)}{(x^2-1)^2}$$

Upravíme závorku.

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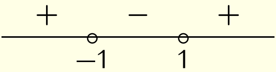
$\boxed{y = \dfrac{x^2+1}{x^2-1}}$ $D(f) = \mathbb{R} \setminus \{-1, 1\}$; průsečík s osou y: $[0, -1]$;

není průsečík s osou x



$$\begin{aligned}
y' &= \frac{(x^2+1)' \cdot (x^2-1) - (x^2+1) \cdot (x^2-1)'}{(x^2-1)^2} \\
&= \frac{2x \cdot (x^2-1) - (x^2+1) \cdot 2x}{(x^2-1)^2} \\
&= \frac{2x \cdot \Big(x^2 - 1 - (x^2+1)\Big)}{(x^2-1)^2} \\
&= \frac{2x\Big(-2\Big)}{(x^2-1)^2} = \frac{-4x}{(x^2-1)^2}
\end{aligned}$$

Dokončíme úpravy

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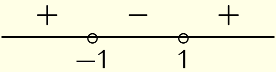
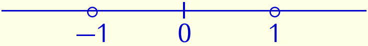
$$\boxed{y = \frac{x^2+1}{x^2-1}}$$ $D(f) = \mathbb{R} \setminus \{-1, 1\}$; průsečík s osou y: $[0, -1]$;

není průsečík s osou x



$$y' = \frac{-4x}{(x^2-1)^2}$$



Derivace je nula pro $x = 0$, což je jediný stacionární bod. Vyneseme tento stacionární bod a body nespojitosti na reálnou osu.

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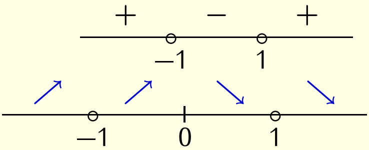
$$\boxed{y = \frac{x^2+1}{x^2-1}}$$

$D(f) = \mathbb{R} \setminus \{-1, 1\}$; průsečík s osou y: $[0, -1]$;

není průsečík s osou x

$$y' = \frac{-4x}{(x^2-1)^2}$$



- Jmenovatel zlomku je pořád nezáporný (jedná se o sudou mocninu). O znaménku tedy rozhoduje pouze čitatel zlomku.
- Protože v čitateli je $(-4x)$, má derivace přesně opačné znaménko jako proměnná x.

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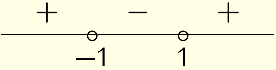
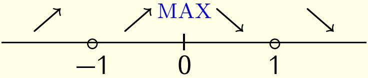
$$\boxed{y = \frac{x^2+1}{x^2-1}}$$

$D(f) = \mathbb{R} \setminus \{-1, 1\}$; průsečík s osou y: $[0, -1]$;

není průsečík s osou x



$$y' = \frac{-4x}{(x^2-1)^2}$$



V bodě $x = 0$ má funkce lokální maximum. Funkční hodnota v tomto bodě je $y(0) = -1$ (bylo počítáno jako průsečík s osou y).

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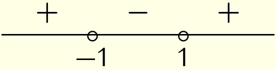
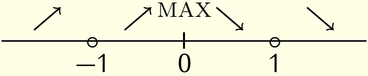
$$\boxed{y=\frac{x^2+1}{x^2-1}}$$ $D(f)=\mathbb{R}\setminus\{-1,1\}$; průsečík s osou y: $[0,-1]$;

není průsečík s osou x



$$y'=\frac{-4x}{(x^2-1)^2}$$



$$y''=-4\cdot\frac{1\cdot(x^2-1)^2-x\cdot 2(x^2-1)\cdot 2x}{(x^2-1)^4}$$

Budeme hledat druhou derivaci. Derivujeme podíl

$$y'=-4\cdot\frac{x}{(x^2-1)^2}$$

podle vzorce

$$\left(\frac{u}{v}\right)'=\frac{u'\cdot v-u\cdot v'}{v^2}.$$

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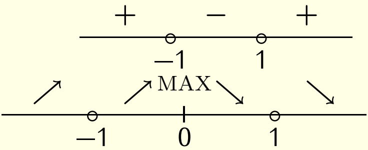
$$\boxed{y=\frac{x^2+1}{x^2-1}}$$ $D(f)=\mathbb{R}\setminus\{-1,1\}$; průsečík s osou y: $[0,-1]$;

není průsečík s osou x



$$y'=\frac{-4x}{(x^2-1)^2}$$

$$y''=-4\cdot\frac{1\cdot(x^2-1)^2-x\cdot 2(x^2-1)\cdot 2x}{(x^2-1)^4}$$

$$=-4\cdot\frac{(x^2-1)\cdot\left(x^2-1-4x^2\right)}{(x^2-1)^4}$$

Protože jsme výraz $(x^2-1)^2$ derivovali jako složenou funkci, nezbavili jsme se možnosti vytknout v čitateli a poté zkrátit.

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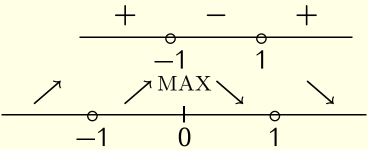
$$\boxed{y=\frac{x^2+1}{x^2-1}}$$ $D(f)=\mathbb{R}\setminus\{-1,1\}$; průsečík s osou y: $[0,-1]$;

není průsečík s osou x



$$y'=\frac{-4x}{(x^2-1)^2}$$

$$y''=-4\cdot\frac{1\cdot(x^2-1)^2-x\cdot 2(x^2-1)\cdot 2x}{(x^2-1)^4}$$

$$=-4\cdot\frac{(x^2-1)\cdot\left(x^2-1-4x^2\right)}{(x^2-1)^4}$$

$$=-4\cdot\frac{-3x^2-1}{(x^2-1)^3}$$

Provedeme krácení a upravíme výraz v závorce.

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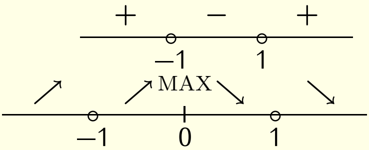
$$\boxed{y=\frac{x^2+1}{x^2-1}}$$ $D(f)=\mathbb{R}\setminus\{-1,1\}$; průsečík s osou y: $[0,-1]$;

není průsečík s osou x



$$y'=\frac{-4x}{(x^2-1)^2}$$

$$y''=-4\cdot\frac{1\cdot(x^2-1)^2-x\cdot 2(x^2-1)\cdot 2x}{(x^2-1)^4}$$
$$=-4\cdot\frac{(x^2-1)\cdot\left(x^2-1-4x^2\right)}{(x^2-1)^4}$$
$$=-4\cdot\frac{-3x^2-1}{(x^2-1)^3}=4\cdot\frac{3x^2+1}{(x^2-1)^3}$$

Dokončíme úpravy.

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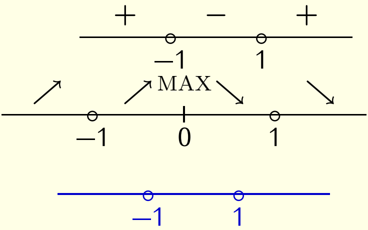
$$\boxed{y=\frac{x^2+1}{x^2-1}}$$

$D(f)=\mathbb{R}\setminus\{-1,1\}$; průsečík s osou y: $[0,-1]$;

není průsečík s osou x

$$y'=\frac{-4x}{(x^2-1)^2}$$

$$y''=4\cdot\frac{3x^2+1}{(x^2-1)^3}$$



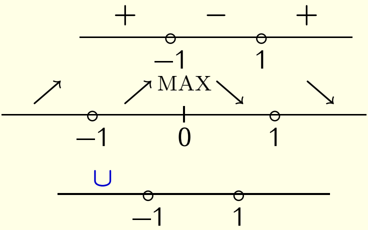
- Druhá derivace není nikdy nulová, protože rovnice $(3x^2+1)=0$ nemá řešení v oboru reálných čísel.
- Znaménko derivace se může změnit nejvýše skokem v bodě nespojitosti. Vyneseme na reálnou osu body nespojitosti.

$$\boxed{y=\frac{x^2+1}{x^2-1}}$$ $D(f)=\mathbb{R}\setminus\{-1,1\}$; průsečík s osou y: $[0,-1]$;

není průsečík s osou x

$$y'=\frac{-4x}{(x^2-1)^2}$$

$$y''=4\cdot\frac{3x^2+1}{(x^2-1)^3}$$



Funkce je konvexní na intervalu $(-\infty,-1)$, protože číslo (-2) leží v tomto intervalu a

$$y''(-2)=4\cdot\frac{\text{kladný výraz}}{[(-2)^2-1]^3}>0$$

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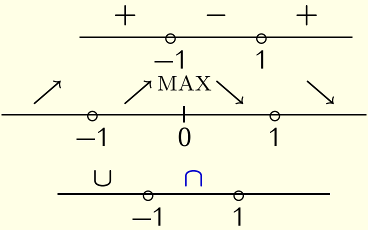
$$\boxed{y=\frac{x^2+1}{x^2-1}}$$

$D(f)=\mathbb{R}\setminus\{-1,1\}$; průsečík s osou y: $[0,-1]$;

není průsečík s osou x

$$y'=\frac{-4x}{(x^2-1)^2}$$

$$y''=4\cdot\frac{3x^2+1}{(x^2-1)^3}$$



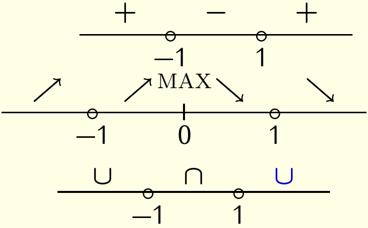
Funkce je konkávní na intervalu $(-1,1)$, protože číslo 0 leží v tomto intervalu a funkce je v tomto bodě nutně konkávní (je zde stacionární bod a lokální maximum – funkce je pod tečnou).

$$\boxed{y=\frac{x^2+1}{x^2-1}}$$ $D(f)=\mathbb{R}\setminus\{-1,1\}$; průsečík s osou y: $[0,-1]$;

není průsečík s osou x

$$y'=\frac{-4x}{(x^2-1)^2}$$

$$y''=4\cdot\frac{3x^2+1}{(x^2-1)^3}$$



Funkce je konvexní na intervalu $(1,\infty)$, protože číslo 2 leží v tomto intervalu a

$$y''(2)=4\cdot\frac{\text{kladný výraz}}{(2^2-1)^3}>0$$

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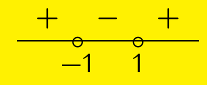

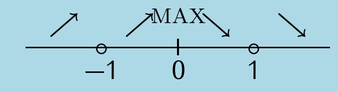

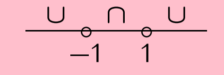

$f(0) = -1;$ $\quad f(\pm\infty) = 1;$ $\quad f(-1\pm) = \mp\infty;$ $\quad f(1\pm) = \pm\infty$

Shrneme nejdůležitější výsledky.

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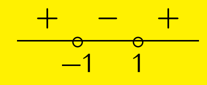
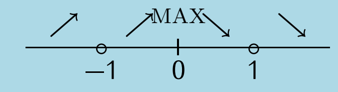
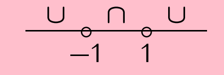
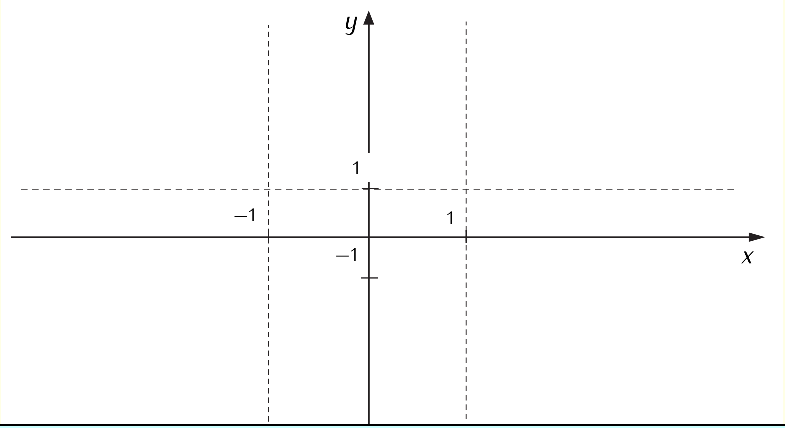






$f(0) = -1;$ $f(\pm\infty) = 1;$ $f(-1\pm) = \mp\infty;$ $f(1\pm) = \pm\infty$



Zakreslíme soustavu souřadnic a asymptoty.

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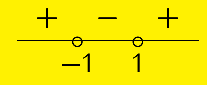
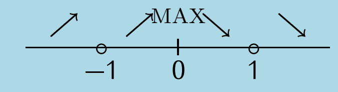
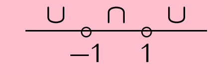
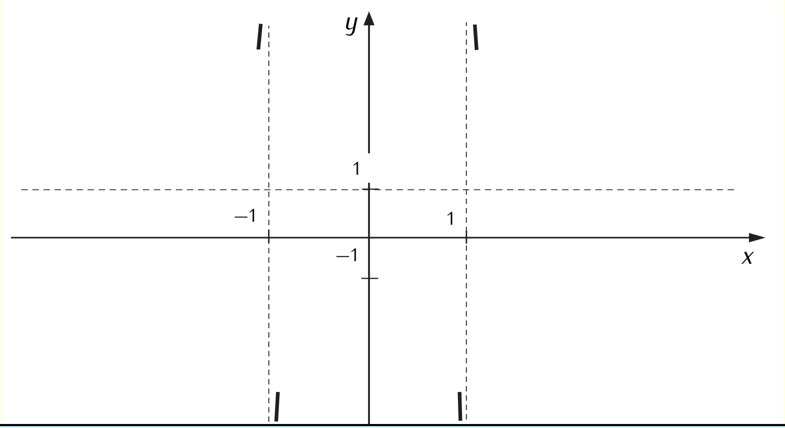






$f(0) = -1$; $f(\pm\infty) = 1$; $f(-1\pm) = \mp\infty$; $f(1\pm) = \pm\infty$



Načrtneme funkci v okolí svislých asymptot. Využijeme monotonie.

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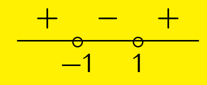
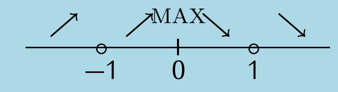
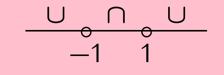
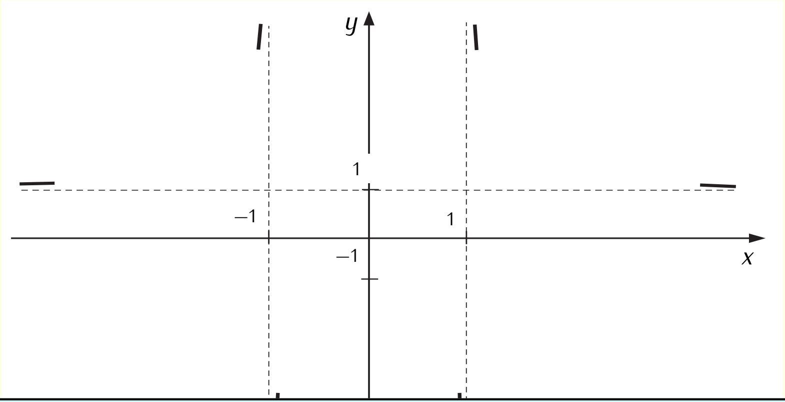






$f(0) = -1;$ $f(\pm\infty) = 1;$ $f(-1\pm) = \mp\infty;$ $f(1\pm) = \pm\infty$



Načrtneme funkci v okolí vodorovné asymptoty. Opět využijeme schema s monotonií.

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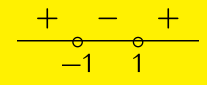
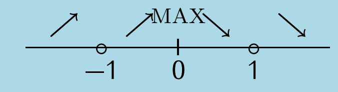
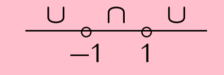
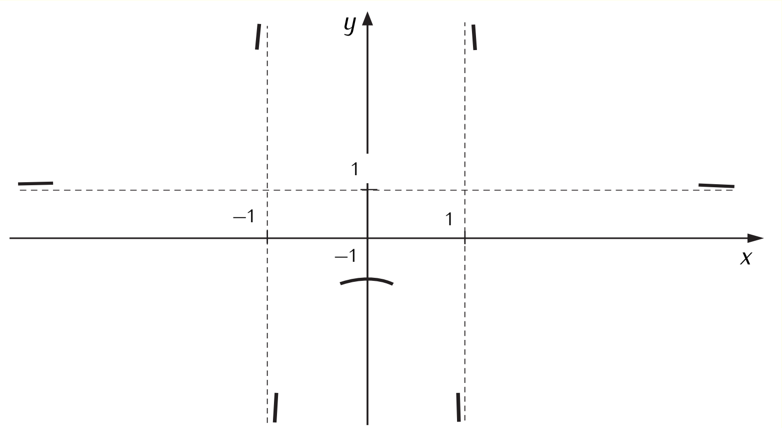






$f(0) = -1;$ $f(\pm\infty) = 1;$ $f(-1\pm) = \mp\infty;$ $f(1\pm) = \pm\infty$



Zakreslíme lokální maximum.

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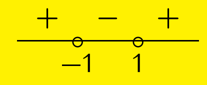
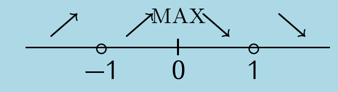
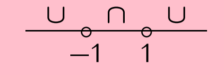
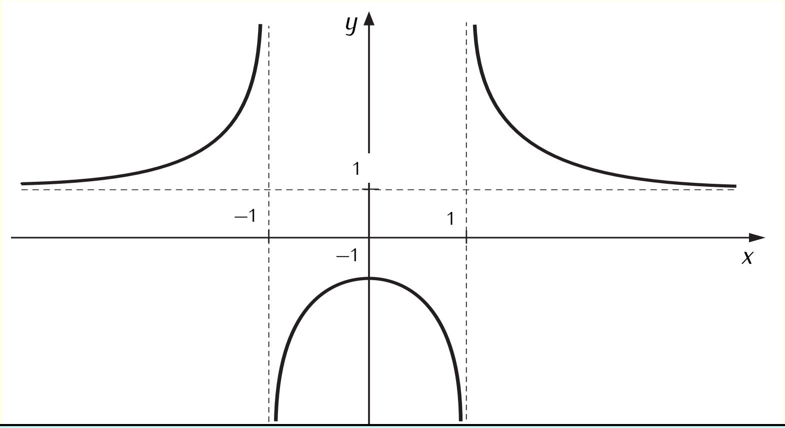






$f(0) = -1$; $f(\pm\infty) = 1$; $f(-1\pm) = \mp\infty$; $f(1\pm) = \pm\infty$



Dokreslíme celý graf.

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KONEC

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