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Apollonius by Inversion

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How many circles can be drawn tangent to three given circles lying in the Euclidean plane? What additional possibilities are there if one or more of the circles are allowed to degenerate into a line or a point? These questions arose out of a problem posed and solved by Apollonius of Perga in the third century B. C.:

> *Given three objects, each of which may be a point, a line, or a circle, construct a circle which passes through each of the points and is tangent to the given lines and circles.*

It is not known exactly how Apollonius solved this problem. Viète, working on hints coming from Pappus, proposed a restoration of the lost Apollonian construction. A later attempt at a restoration can be found in [7]. The various aspects of the problem have been a challenge to generations of geometers; Descartes, Newton, Euler, Gauss, and Cauchy are among the names occurring in its history. In the nineteenth century it served as a test case in the competition between rival schools of geometry. Elegant solutions eventually became common in textbooks (see, for example, [**8**] or [**10**]). Yet, according to N. A. Court in his historical survey [**1**], until the last quarter of the nineteenth century no one was particularly concerned with the number of solutions that the problem may have, which is the question that we wish to address.

In 1898 R. F. Muirhead [**9**] made the first noteworthy attempt at an exhaustive enumeration of the cases that can arise. We shall indicate here how his methods, although skillful, led to incomplete and unnecessarily complicated results. More recently, J. M. Fitz-Gerald [**6**] discussed the problem with special emphasis on the cases in which the three given objects are lines and circles with the property that no point is common to all three. In the present treatment we achieve an explicit listing of canonical forms for all the possible cases. Our work also serves as an independent proof of the main result in [**11**] to the effect that no specialization of the problem can produce exactly seven circles tangent to the given circles.

The classification of configurations of three given objects

The natural setting for an economical classification of cases for the problem is the inversive plane, which we shall think of as the Euclidean plane completed by a single point at infinity, P_∞. (See [**2**, pp. 77–95] or [**5**, pp. 103–131] for definitions and elementary theorems.) Since a line is simply an inversive circle which passes through P_∞, it is convenient to refer to all circles and lines as inversive circles, or ***i*-circles** for short. We use the word **object** as defined in the problem, that is, an object can be a line, circle, or point.

Since the incidence properties of objects are not altered by inversions, any enumeration of cases should be reduced to those which cannot be transformed into one another by a product of inversions. Muirhead failed to do this and therefore enumerated many more cases than necessary. In order to simplify our enumeration we use the following theorems of inversive geometry [**2**, § 6.5].

I. *A pair of disjoint i-circles can be inverted to a pair of concentric circles.*
II. *A pair of intersecting i-circles can be inverted to a pair of intersecting lines.*
III. *A pair of tangent i-circles can be inverted to a pair of parallel lines.*

In any case where two or more of these theorems apply we shall use the relevant theorem that appears first on our list. For instance, two intersecting circles disjoint from the third will be inverted to a pair of concentric circles and a circle intersecting none of them (rather than a pair of intersecting lines and a circle intersecting neither). This convention permits us to seek canonical forms in the following list of disjoint and exhaustive categories of three given objects.

1. Three points.
2. Two points and a line.
3. One point and two concentric circles.
4. One point and two intersecting lines.
5. One point and two parallel lines.
6. Two concentric circles and one *i*-circle.
7. Two intersecting lines and one *i*-circle meeting both lines.
8. Two parallel lines and one *i*-circle tangent to both lines.

For example, a triple of objects consisting of a point, line, and circle would be included under 3, 4, or 5 above depending on whether the line missed the circle, was secant to the circle, or was tangent to the circle. In our table of solutions to the problem of Apollonius, we make one exception to our stated convention: when the given point in 5 is P_∞, we picture this case as two circles tangent at the given point (so that this point appears in the picture).

It is *not* true that any two configurations (given triples of objects) with the same canonical form are inversively equivalent. This follows from the fact that the ratio of the radii of two concentric circles and the angle between two intersecting lines are inversive invariants [**3**]. However, the values of these parameters do not affect the *number* of tangent objects which are solutions to the problem for a given configuration, which is what we seek. This number depends only on the incidence and separation properties of the given objects. As our concluding remarks will indicate, it is an amusing exercise to determine this number by an inspection of the canonical form of the given configuration.

Sketches of the 33 possible canonical forms, 15 involving points and 18 involving only *i*-circles, are arranged in TABLE 1 and TABLE 2 respectively. The columns of these tables are labeled to indicate the number of solutions, where a solution is any object tangent to the three given objects. With Muirhead, we regard any object as being self-tangent. Thus, for example, a configuration of three mutually tangent *i*-circles with no point common to all three (category 8 in our list) gives rise to five solutions and not just two. In TABLES 1 and 2, the numbers under a diagram refer to Muirhead's classification [**9**, Table IV and Figures 50–114]. Note that his classification is incomplete as well as redundant.

We also find it convenient to include in TABLE 2 the descriptive labels of Fitz-Gerald who uses the symbol I for each intersecting pair of given *i*-circles, the symbol T for each tangent pair, and the symbol S for a given *i*-circle which separates the other two. As he points out [**6**, p. 18], it is unnecessary to specify which circles are involved since we are concerned only with the inversive relationship between the given *i*-circles taken in pairs and (should none of the circles intersect) whether or not one of the circles separates the other two. When the given *i*-circles have a point in common we put a bracket around the Fitz-Gerald label. Thus the appropriate label for three mutually tangent *i*-circles is either TTT (no point in common) or $[TTT]$ (a point in common). We note that almost all of the cases may be distinguished by their labels; the exceptions are III and $[III]$ and here we resort to subscripts.

We invite the reader to reproduce our 33 canonical forms of given configurations by systematically examining the eight categories listed above. As a typical example, here is how one might argue that category 7 leads to seven canonical forms. Begin with a pair of lines that meet in a point O (and, of course, in P_∞). The third given *i*-circle, which we'll call γ, can (a) intersect both lines, (b) intersect one line and be tangent to the other, or (c) be tangent to both. The *i*-circle γ cannot be disjoint from either line since that possibility belongs to category 6. One must now investigate each subcase in turn.

(a) The *i*-circle γ can intersect both lines in three distinct ways. First, γ can pass through the two points of intersection of the lines; this is $[III]_2$. Or, γ can pass through exactly one of the points of intersection; this is $[III]_1$. (Note that in TABLE 2 we show γ passing through P_∞ but we could equally well have shown γ passing through O.) Finally, γ can miss both O and P_∞; the two subcases are γ separates O from P_∞, which is III_2, or γ doesn't separate O from P_∞, which is III_1.

(b) The *i*-circle γ can be tangent to just one of the lines in two distinct ways. First, γ is tangent at P_∞ (or equivalently at O); this is $[IIT]$. Otherwise γ is tangent at some other point of one of the lines; this is IIT.

(c) There is essentially only one way for a circle to be tangent to both lines of an intersecting pair, namely ITT.

It is possible to adopt various other conventions for enumerating solutions; for example, one may argue that neither a point nor a given object should be regarded as a solution. However, it is clear from the tables that no convention would give rise to a case of seven tangent objects.

There appears to be some confusion in the literature as to what constitutes the "general case" of the problem. We may reasonably regard three *i*-circles to be in general position if they have *no common point* and if *no two are tangent*. The relevant cases are then $\varnothing$, III_1, III_2, II, I, and S; and our results are consistent with the statement attributed to Sturm [**6**], [**11**] that the number of circles tangent to three circles in general position is either 8, 4, or 0.

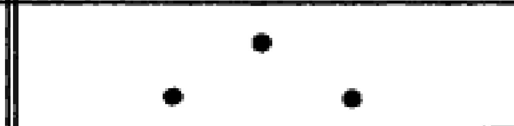
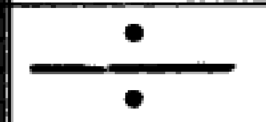
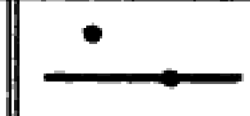
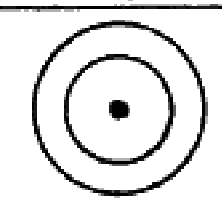
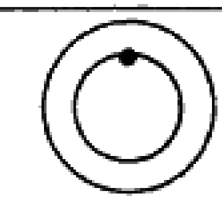
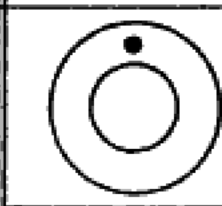
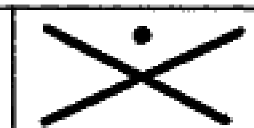
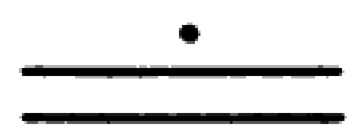
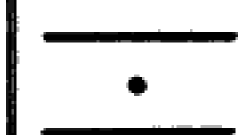
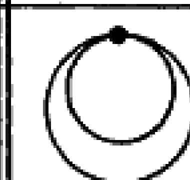
CONFIGURATION	NUMBER OF SOLUTIONS 0	1	2	3	4	∞
1. Three points.		 *				
2. Two points and a line.	 114, *	 # ;  #	 114, *			
3. One point and two concentric circles.	 50, 52, *		 #		 50, 52, *	
4. One point and two intersecting lines.		 #	 # ;  51, *			
5. One point and two parallel lines.		 #	 #	 #		 #

TABLE 1. **The fifteen canonical forms in which at least one of the given objects is a point. Numbers below a configuration refer to Figures 50–114 in Muirhead's classification [9]; an asterisk refers to his Table IV. The symbol # indicates that the configuration is not considered in [9].**

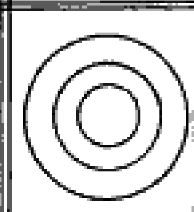
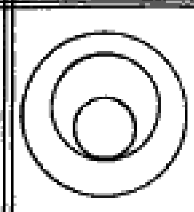
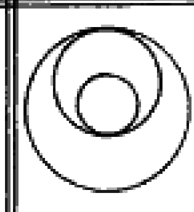
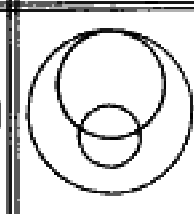
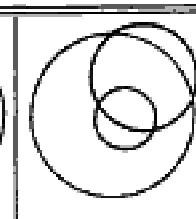
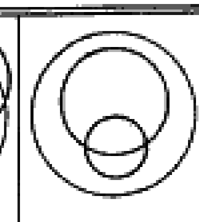
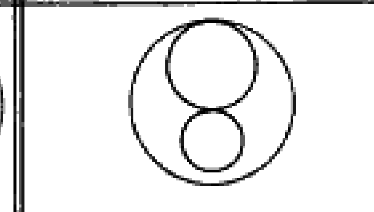
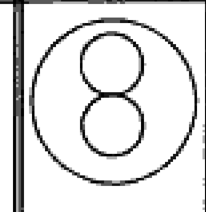
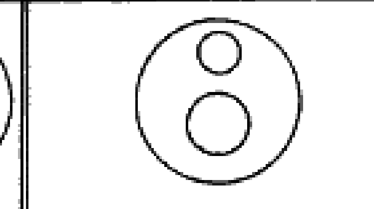
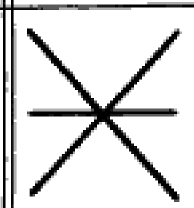
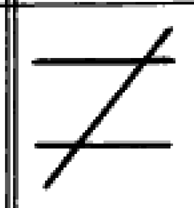
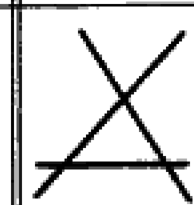
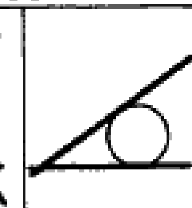
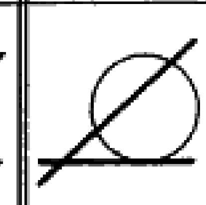
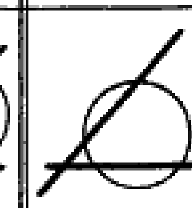
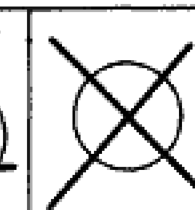
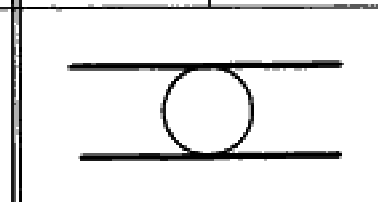
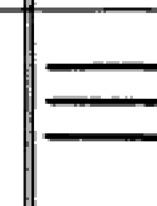
CONFIGURATION	NUMBER OF SOLUTIONS: 0	2	3	4			5		6	8		∞
6. Two concentric circles and one *i*-circle.	 57, 58 103, 104 *S*	 71–74 *ST*	 92 *STT*	 76–81 *IT*	 63, 64 108–110 *II*	 59–62 105–107 *I*	 89, 91 *TT*		 69, 70, 75 *T*	 55, 56, 102 ∅		
7. Two intersecting lines and one *i*-circle meeting both.		 99 $[III]_2$	 97, 98 [*IIT*]				 86–88 $[III]_1$	 93–95 *ITT*	 82–85 *IIT*	 65–67 111, 113 III_1	 68, 112 III_2	
8. Two parallel lines and one *i*-circle tangent to both.							 # *TTT*					 100, 101 [*TTT*]

TABLE 2. **The eighteen canonical forms in which all three given objects are *i*-circles. Numbers below a configuration refer to Figures 50–114 in Muirhead's classification [9]; the symbol # indicates the configuration was not considered in [9]. The letter symbols under each configuration are adapted from Fitz-Gerald's descriptive labels [6].**

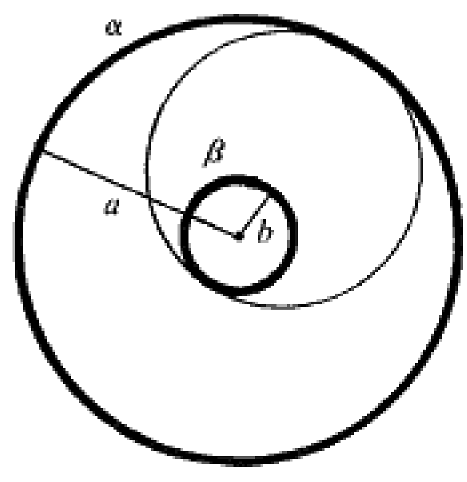


FIGURE 1. The solution circles of radius $(a+b)/2$ for categories 3 and 6.

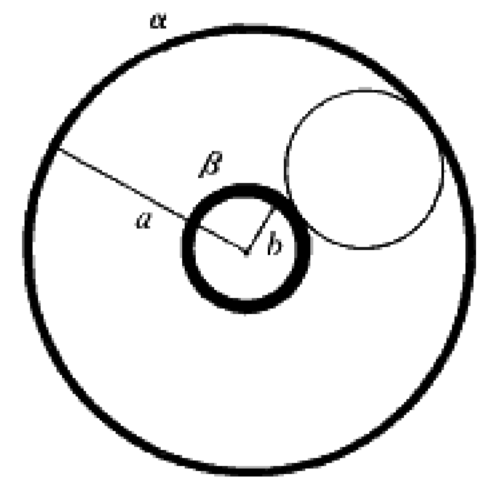


FIGURE 2. The solution circles of radius $(a-b)/2$ for categories 3 and 6.

Counting the solutions to the problem

When all possible configurations of three given objects have been enumerated, we can address the question of counting the number of objects tangent to the three given ones in each canonical arrangement. We shall indicate how one might investigate those cases in which a pair of the given objects are i-circles, say α and β.

H. S. M. Coxeter [**4**, section 5] has illustrated the process by a careful discussion of the two possibilities denoted by S and $\varnothing$ in TABLE 2, category 6; these arise when the three given objects are disjoint i-circles. Actually, his argument applies to all canonical forms that fall into our category 3 or 6. These are the cases in which α and β are concentric circles whose radii satisfy $a>b$. Since α and β are concentric, the circles that touch both consist of two families of congruent circles in the closed annulus bounded by α and β: one family having radius $(a+b)/2$ (FIGURE 1) and the other having radius $(a-b)/2$ (FIGURE 2). One can imagine a potential solution circle rolling about the annulus like a hoop until it reaches a position where it touches the third given object.

An analogous argument is valid for those canonical forms in which α and β are parallel lines (as in category 5 or 8): the solution circles (that are disjoint from P_∞) are then congruent circles touching both α and β.

Finally, α and β could be a pair of intersecting lines as in category 4 or 7. We conclude this discussion by giving a detailed analysis of the particular case *IIT* shown in FIGURE 3. In this case it is clear that any i-circle tangent to the intersecting lines α and β must be a circle in one of the quadrants A, B, C, or D into which they divide the plane. A general circle of this sort may be thought of as belonging to a family of circles whose members grow continuously from very small circles near the point O to very large circles far out in their quadrant. FIGURE 4 shows the six "stages of growth" at which the general circles of quadrants B, C, and D become tangent to circle γ. FIGURE 5 is the image of FIGURE 4 under inversion with centre P. It serves as a check on our count of tangent i-circles.

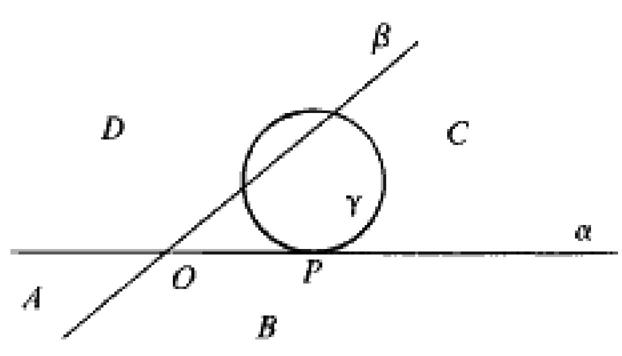


FIGURE 3. The canonical form for the case *IIT*.

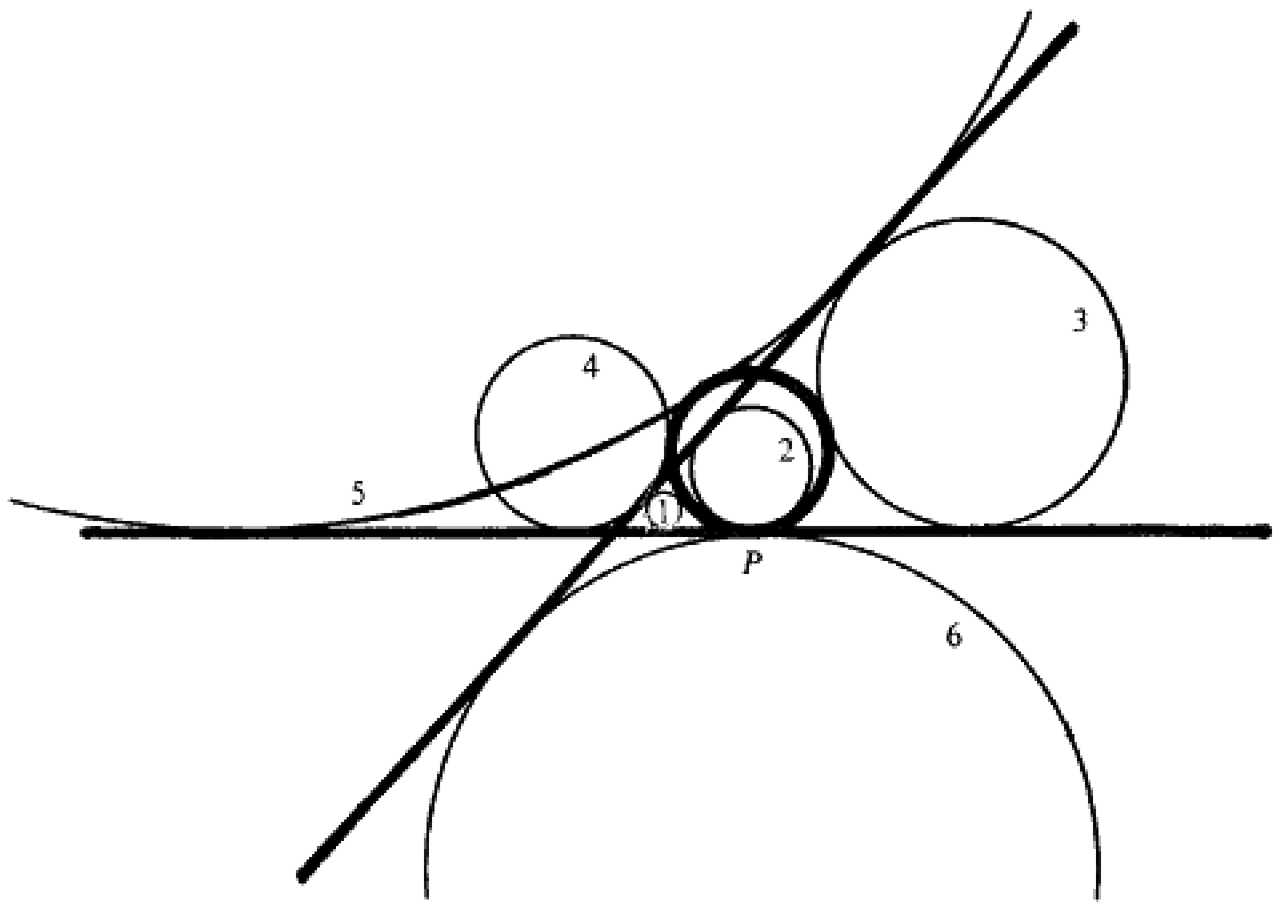


FIGURE 4. **Counting the solutions for the case *IIT*. The three given objects are drawn with heavy line.**

Once again we remind the reader that this paper calls for participation. Half the fun lies in reproducing the tables and this is a two-step project. First one has to list the 33 essentially different cases of the problem. Then in each of these cases one has to count solutions.

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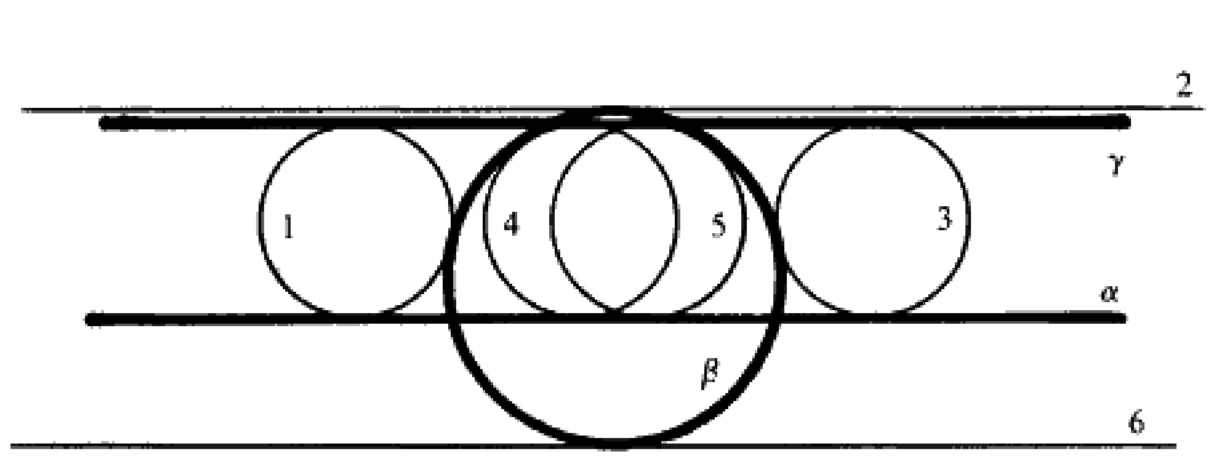

FIGURE 5. **Alternative form of the case *IIT*.**